From the course notes

"Classical Groups and Geometry, version of 7 April 2015"

(that is, **CGG**), the following problems:

CGG PROBLEMS (10.36), (10.37), (10.38), (10.39)

5. Let t be an indeterminate. Let $L = \{\sum_{i=h}^{\infty} p_i t^i \mid h \in \mathbb{Z}, p_i \in \mathbb{C}\}$. (Notice that negative integers h are allowed.) We make the standard identifications $\mathbb{R} \leq \mathbb{C} = \mathbb{C}t^0$.

For $p = \sum_{i=h}^{\infty} p_i t^i \in L$, define the *minimal degree*, $\mathrm{md}(p)$, to be the smallest i with $p_i \neq 0$. (Note: $\mathrm{md}(0) = \infty$.)

Observe that $\{p \in L \mid \mathrm{md}(p) \geq 0\} = \mathbb{C}[[t]]$, the ring of formal power series in t. L is then the field of Laurent series, the quotient field of $\mathbb{C}[[t]]$ (because, for p with $\mathrm{md}(p) > 0$, $(1-p)^{-1} = \sum_0^{\infty} p^i$ is well-defined and in L).

The usual complex conjugation in $\mathbb{C}$ (denoted by $c \mapsto \bar{c}$) extends to an involutory automorphism σ on all L:

$$(\textstyle\sum_i p_i t^i)^{\sigma} = \sum_i \bar{p}_i t^i \,.$$

Let $\vec{e}_1, \ldots, \vec{e}_n$ be a basis for the L-space $V = L^n$ (for $n \geq 2$), and let f be the σ-hermitian form on V given by

$$f(\textstyle\sum_{i=1}^{n} x_i \vec{e}_i, \sum_{i=1}^{n} y_i \vec{e}_i) = \sum_{i=1}^{n} x_i y_i^{\sigma} \,,$$

so that the associated Gram matrix is the $n \times n$ identity matrix.

We then have the unitary group

$$\begin{aligned} \mathrm{GU}(V,f) &= \{G \in \mathrm{GL}_n(L) \mid f(x,y) = f(xG, yG), \text{ all } x, y \in V\} \\ &= \{G \in \mathrm{GL}_n(L) \mid G(G^{\sigma})^{\top} = I\} \,. \end{aligned}$$

For $G = (g_{i,j})_{i,j} \in \mathrm{Mat}_n(L)$, let $\mathrm{md}(G) = \min_{i,j}(\mathrm{md}(g_{i,j}))$. For integral $k \geq 0$, let $U_k = \{G \in \mathrm{GU}(V,f) \mid \mathrm{md}(I - G) \geq k\}$.

(a) Prove that $f(\vec{x}, \vec{x}) = 0$ if and only if $\vec{x} = \vec{0}$.
(b) For $G \in \mathrm{GU}(V,f)$, prove that $\mathrm{md}(G) \geq 0$, hence $\mathrm{GU}(V,f) = U_0$.
(c) Prove that U_k is normal in $\mathrm{GU}(V,f)$, for all $k \geq 0$.
(d) Prove that, for $k \neq m$, $U_k \neq U_m$, and that $\bigcap_k U_k = 1$.
(e) Prove that, for all $k \geq 1$, U_k / U_{k+1} is abelian.