A Euclidean Algorithm example

We now calculate $\gcd(z^6, 9z^5+8z^4+2z^3+7z^2+6) = 1$ over $\mathbb{F}_{11}$ using the Euclidean Algorithm.

At Step i we define $q_i(z)$, $r_i(z)$, $s_i(z)$, and $t_i(z)$ using

$$\begin{aligned} r_{i-2}(z) &= q_i(z)r_{i-1}(z) + r_i(z) \\ s_i(z) &= s_{i-2}(z) - q_i(z)s_{i-1}(z) \\ t_i(z) &= t_{i-2}(z) - q_i(z)t_{i-1}(z) \ . \end{aligned}$$

Step i	$q_i(z)$	$r_i(z)$	$s_i(z)$	$t_i(z)$
-1	$-$	z^6	1	0
0	$-$	$9z^5+8z^4+2z^3+7z^2+6$	0	1
1	$5z+9$	$6z^4+2z^3+3z^2+3z+1$	1	$6z+2$
2	$7z+10$	$5z^3+7z+7$	$4z+1$	$2z^2+3z+3$
3	$10z+7$	$\mathbf{10z^2+5z+7}$	$4z^2+6z+5$	$\mathbf{2z^3+10z+3}$
4	$6z+8$	$2z+6$	$9z^3+9z^2+$ $+3z+5$	$10z^4+6z^3+8z^2+$ $+4z+1$
5	$5z+4$	5	$10z^4+7z^3+8z^2+$ $+2z+7$	$5z^5+7z^4+4z^3+$ $+3z^2+10$
6	$7z+10$	0	$-$	$-$

Thus

$$5 = (10z^4+7z^3+8z^2+2z+7)z^6 + (5z^5+7z^4+4z^3+3z^2+10)(9z^5+8z^4+2z^3+7z^2+6)$$

and, after dividing by 5 (that is, multiplying by $5^{-1} = 9$), we have

$$\begin{aligned} 1 &= \gcd(z^6, 9z^5+8z^4+2z^3+7z^2+6) \\ &= (2z^4+8z^3+6z^2+7z+8)z^6 + \\ &\qquad +(z^5+8z^4+3z^3+5z^2+2)(9z^5+8z^4+2z^3+7z^2+6) \ . \end{aligned}$$

Step 1.

$$\begin{array}{r|ccccccc|l}
 & 5z & +9 & & & & & & = q_1(z) \\ \hline
9z^5+8z^4+2z^3+7z^2 \quad +6 & z^6 & & & & & & & = r_{-1}(z) \\
 & z^6 & +7z^5 & +10z^4 & +2z^3 & & +8z & & \\ \cline{2-7}
 & & 4z^5 & +z^4 & +9z^3 & & +3z & & \\
 & & 4z^5 & +6z^4 & +7z^3 & +8z^2 & & +10 & \\ \cline{3-8}
 & & & 6z^4 & +2z^3 & +3z^2 & +3z & +1 & = r_1(z)
\end{array}$$

$$\begin{aligned} r_{-1}(z) &= q_1(z)r_0(z) + r_1(z) \\ z^6 &= (5z+9)(9z^5+8z^4+2z^3+7z^2+6) + (6z^4+2z^3+3z^2+3z+1) \\ q_1(z) &= 5z+9 \\ r_1(z) &= 6z^4+2z^3+3z^2+3z+1 \end{aligned}$$

$$\begin{aligned} s_1(z) &= s_{-1}(z) - q_1(z)s_0(z) \\ s_1(z) &= 1 - (5z+9)0 = 1 \end{aligned}$$

$$\begin{aligned} t_1(z) &= t_{-1}(z) - q_1(z)t_0(z) \\ t_1(z) &= 0 - (5z+9)1 = 6z+2 \end{aligned}$$

Step 2.

$$
\begin{array}{r|rrrrrrl}
 & 7z & +10 & & & & & = q_2(z) \\
\cline{2-7}
6z^4+2z^3+3z^2+3z \quad +1 & 9z^5 & +8z^4 & +2z^3 & +7z^2 & & +6 & = r_0(z) \\
 & 9z^5 & +3z^4 & +10z^3 & +10z^2 & +7z & & \\
\cline{2-6}
 & & 5z^4 & +3z^3 & +8z^2 & +4z & +6 & \\
 & & 5z^4 & +9z^3 & +8z^2 & +8z & +10 & \\
\cline{3-7}
 & & & 5z^3 & & +7z & +7 & = r_2(z)
\end{array}
$$

$$
\begin{aligned}
r_0(z) &= q_2(z)r_1(z) + r_2(z) \\
9z^5+8z^4+2z^3+7z^2+6 &= (7z+10)(6z^4+2z^3+3z^2+3z+1) + (5z^3+7z+7) \\
q_2(z) &= 7z+10 \\
r_2(z) &= 5z^3+7z+7
\end{aligned}
$$

$$
\begin{aligned}
s_2(z) &= s_0(z) - q_2(z)s_1(z) \\
s_2(z) &= 0 - (7z+10)1 = 4z+1
\end{aligned}
$$

$$
\begin{aligned}
t_2(z) &= t_0(z) - q_2(z)t_1(z) \\
t_2(z) &= 1 - (7z+10)(6z+2) = 2z^2+3z+3
\end{aligned}
$$

Step 3.

$$
\begin{array}{r|rrrrrl}
 & 10z & +7 & & & & = q_3(z) \\
\cline{2-6}
5z^3+ \qquad +7z \quad +7 & 6z^4 & +2z^3 & +3z^2 & +3z & +1 & = r_1(z) \\
 & 6z^4 & & +4z^2 & +4z & & \\
\cline{2-5}
 & & 2z^3 & +10z^2 & +10z & +1 & \\
 & & 2z^3 & & +5z & +5 & \\
\cline{3-6}
 & & & 10z^2 & +5z & +7 & = r_3(z)
\end{array}
$$

$$
\begin{aligned}
r_1(z) &= q_3(z)r_2(z) + r_3(z) \\
6z^4+2z^3+3z^2+3z+1 &= (10z+7)(5z^3+7z+7) + (10z^2+5z+7) \\
q_3(z) &= 10z+7 \\
r_3(z) &= 10z^2+5z+7
\end{aligned}
$$

$$
\begin{aligned}
s_3(z) &= s_1(z) - q_3(z)s_2(z) \\
s_3(z) &= 1 - (10z+7)(4z+1) = 4z^2+6z+5
\end{aligned}
$$

$$
\begin{aligned}
t_3(z) &= t_1(z) - q_3(z)t_2(z) \\
t_3(z) &= (6z+2) - (10z+7)(2z^2+3z+3) \\
&= (6z+2) + (z+4)(2z^2+3z+3) \\
&= 2z^3+10z+3
\end{aligned}
$$

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