All answers must be justified appropriately.

1. (RANDOMNESS AND THE SECOND HARMONIC) Let $\mathbf{x} = (x_1, \ldots, x_n) \in \pm 1^n$ be a ± 1-sequence of length n.

Consider a fixed position k, with $1 \leq k \leq n$. The *second harmonic* at k, a random variable SH_k on $\pm 1^n$, is given by

$$SH_k(\mathbf{x}) = \sum_{i=1}^{n} x_{k-i} x_{k+i} \,,$$

where, in order for this to make sense, we set $x_j = 0$ for $j < 1$ and $j > n$. For example, with $n = 9$ we have

$$SH_3(\mathbf{x}) = x_2 x_4 + x_1 x_5$$

and

$$SH_6(\mathbf{x}) = x_5 x_7 + x_4 x_8 + x_3 x_9 \,.$$

Since each SH_k is a quadratic form, we saw in class that, for all n and all k, we have $\mathbf{E}(SH_k) = 0$.

(a) Prove that, if $n > 2$, then there is no sequence $\mathbf{x} \in \pm 1^n$ with $SH_k(\mathbf{x}) = 0$, for all k. (That is, SH-perfect sequences do not exist.)

(b) Call the sequence $\mathbf{x}$ *SH-optimal* if $|SH_k(\mathbf{x})| \leq 1$, for all k. Prove that the length n sequence

$$(+1 + 1 \;\; -1 - 1 \;\; +1 + 1 \;\; -1 - 1 \;\; +1 + 1 \;\; -1 - 1 \;\; \ldots)$$

(alternating $+1 + 1$ and $-1 - 1$) is SH-optimal.

REMARK. In particular, pseudorandom sequences might not be very random at all. This problem actually came up in some work done by chemists here at MSU.

2. (k-TUPLE EXPECTATION FOR THE BIASED COIN) Consider $\{H, T\}^n$, length n Bernoulli sequences w of n flips with a biased coin. ($\mathbf{P}(H) = p$, $\mathbf{P}(T) = q$)

For the k-tuple $\mathbf{x} \in \{H, T\}^k$, let the random variable $K^{\mathbf{x}}$ evaluated at the sequence $w \in \{H, T\}^n$ count the number of times that the k-tuple $\mathbf{x}$ appears in consecutive positions in w, read cyclically. For instance,

$$K^{HTT}(THTTHTHT) = 2 \quad \text{because of} \quad T\mathbf{HTT}HTHT \text{ and } \mathbf{T}HTTHT\mathbf{HT}\,.$$

Find, for each k, n, and $\mathbf{x} \in \{H, T\}^k$, the expected value $E(K^{\mathbf{x}})$. (Make sure you justify your answer.)

(HINT: This should involve the function $h(\mathbf{x})$ which counts the number of heads in $\mathbf{x}$.)

3. (SOME DEBRUIJN TYPE SEQUENCES.) For this problem and the next, not only list the sequences but explain why these are the only ones. (Do not just grind these out with a computer.) You need only list one from each cycle class.

Find all binary 8-tuples in which each 2-tuple 00, 01, 10, and 11 appears (cyclically) exactly twice. (This is the 2-tuple perfect property for length 8.)

Recall: *All answers must be justified appropriately.*

4. (Further DeBruijn type sequences.) Following the same instructions as in the previous problem, find all 9-tuples of $\{H, T\}^9$ in which (cyclically) HH appears 4 times, HT appears twice, TH appears twice, and TT appears once.

Remark. This is related to Problem 2 for the biased coin with $p = 2/3$.

Recall: *All answers must be justified appropriately.*