All answers must be justified appropriately.

(1) Let $Y : \Omega \longrightarrow \mathbb{R}$ be a random variable on the probability space Ω. Let $\Omega = \bigcup_{i=1}^{m} \Omega_i$ exhibit Ω as the disjoint union of subsets Ω_i, each having the property that

$$\text{if } a, b \in \Omega_i \text{ then } Y(a) = Y(b)\,.$$

For each i let $\omega_i \in \Omega_i$. Prove that $\mathbf{E}(Y) = \sum_{i=1}^{m} \mathbf{P}(\Omega_i) Y(\omega_i)$.

REMARK. The case $\Omega_i = \{\, \omega \in \Omega \mid Y(\omega) = Y(\omega_i) \,\}$ is essentially our definition of the expected value, while the case $\Omega_i = \{\omega_i\}$ gives the proposition that we proved in class on January 25.

(2) A lemma we proved in class (on January 29) stated tha $\mathbf{E}(S_n) = pn$ and $\mathrm{var}(S_n) = npq$, where $p = \mathbf{P}(H)$, $q = 1 - p = \mathbf{P}(T)$, and S_n counts the number of heads in a Bernoulli n-tuple. The first of these we proved directly, while the second we proved using the method of generating functions.

Now go back and prove the first part of the lemma

$$\mathbf{E}(S_n) = pn$$

using the method of generating functions.

(3) (THE MONTY HALL PROBLEM) Contestants on "Let's Make a Deal" are confronted with three closed doors. Behind one of the doors is a prize of \$999,999, while behind the other two doors there is nothing.

The rules of the game are: the contestant first chooses one door but leaves it closed. Host Monty Hall[1] (who knows where the money is) then goes to one of the two doors not selected by the contestant and opens it, demonstrating that it has nothing behind it. The contestant now has a choice to make: the contestant either may keep (and open) the originally chosen door or may switch to (and open) the other closed door. The contestant wins if the opened door has the money behind it. (So, for instance, the contestant picks door 1; Monty opens door 3, which does not have the money behind it; the contestant must then either keep with 1 or switch to 2. If the contestant keeps 1, the contestant wins if the money is behind 1 and loses if the money is behind 2. If the contestant switches to 2, then the contestant wins if the money is behind 2 and loses if the money is behind 1.)

Consider the probability space Ω consisting of the nine pairs (i, j) with $1 \le i \le 3$ and $1 \le j \le 3$, where i is the door originally selected by the contestant and j is the door hiding the money. Each pair is equally likely (that is, has probability $1/9$).

For this problem, you are to analyze the game. In doing so define two random variables on Ω. The first variable K (for "keep") gives the gain associated with each pair (i, j) if the contestant follows the strategy of keeping the

[1] no relation

original door. The second random variable S (for "switch") gives the gain associated with each pair if the contestant follows the strategy of switching doors. Write down K and S explicitly and then calculate $\mathbf{E}(K)$ and $\mathbf{E}(S)$.

What is the best strategy?

(4) (Door Prize) At a party, you have the opportunity to win the door prize. There are $n = 2m + 1$ identical boxes, and the prize is in one of the boxes (all the others being empty). Each box is equally likely to contain the prize.

The rules for the drawing are: Someone separates the boxes into two groups of boxes—say, groups of size k and s with $k < s$. After the boxes have been separated into two groups, you get to select one of the two groups. All of the boxes from that group are then yours; if the prize is in one of those boxes, you win. If the prize is in a box from the group you did not select, you lose.

What is your best strategy for winning the prize? Model the drawing by a probability space. Use expectations of random variables to prove that your strategy is the best one.

(5) (Essay question—up to one page.) I claim that the situation of Problem 3 is a special case of that in Problem 4. Discuss this claim.