We find that

$$\det\begin{pmatrix} t & 0 & \dots & 0 & 0 & a_0 \\ -1 & t & \dots & 0 & 0 & a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & t & a_{n-2} \\ 0 & 0 & \dots & 0 & -1 & t+a_{n-1} \end{pmatrix} = \sum_{i=0}^{n} a_i t^i \,,$$

for $a_n = 1$.

PROOF (SKETCH): Proof is by induction on n.

Initialization: for $n = 1$

$$\det(t + a_0) = t + a_0 \,,$$

as claimed.

[Although it is not strictly necessary, it probably is a help to do the $n = 2$ case as well:

$$\begin{pmatrix} t & a_0 \\ -1 & t+a_1 \end{pmatrix} = t(t + a_1) - (-1)a_0 = t^2 + a_1 t + a_0 \,.$$

Doing this ahead of time also suggests the likely general answer.]

Induction step: Assume that the result is true for $n - 1$. We then need to prove the result for n.

We expand the determinant along the first row.

$$\det\begin{pmatrix} t & 0 & 0 & \dots & 0 & 0 & a_0 \\ -1 & t & 0 & \dots & 0 & 0 & a_1 \\ 0 & -1 & t & \dots & 0 & 0 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & t & 0 & a_{n-3} \\ 0 & 0 & 0 & \dots & -1 & t & a_{n-2} \\ 0 & 0 & 0 & \dots & 0 & -1 & t+a_{n-1} \end{pmatrix} =$$

$$t\det\begin{pmatrix} t & 0 & \dots & 0 & a_1 \\ -1 & t & \dots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & t & a_{n-2} \\ 0 & 0 & \dots & -1 & t+a_{n-1} \end{pmatrix} + (-1)^{1+n} a_0 \det\begin{pmatrix} -1 & t & \dots & 0 & 0 \\ 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & t \\ 0 & 0 & \dots & 0 & -1 \end{pmatrix}$$

which, by induction, is

$$t\left(\sum_{j=0}^{n-1} a_{j+1} t^j\right) + (-1)^{1+n} a_0 (-1)^{n-1} = \sum_{i=0}^{n} a_i t^i \,,$$

as desired.