All answers must be justified appropriately.

From TREIL do the following exercises.
(Note that when we refer to Exercise a.b.c, we mean Exercise b.c from Chapter a.)
TREIL page 17: Exercises 1.3.2, 1.3.3
TREIL page 23: Exercises 1.5.1, 1.5.3, 1.5.5, 1.5.6

1. Prove that $\mathbb{P}_2$, the vector space of all real polynomials of degree at most 2, is isomorphic to $\mathbb{R}^3$, the vector space of all real column vectors of length 3.

2. For $\mathbb{F}$-linear transformations
$$T_1\colon V \longrightarrow W \quad \text{and} \quad T_2\colon U \longrightarrow V\,,$$
the composition map $S = T_1T_2\colon U \longrightarrow W$ is given by
$$S(\mathbf{x}) = T_1(T_2(\mathbf{x}))$$
Prove that if T_1 and T_2 are isomorphisms, then S is a isomorphism.

3. (a) Is the map $T : \mathbb{F}^3 \to \mathbb{F}^2$, defined by
$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^2 + 2y \\ z \end{pmatrix}$$
a linear transformation? Justify.

   (b) Is the map $T : \mathbb{F}^3 \to \mathbb{F}^2$, defined by
$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y \\ z \end{pmatrix}$$
a linear transformation? Justify.

   (c) Is the map $T : \mathbb{F}^3 \to \mathbb{F}$ defined by
$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 4 \end{pmatrix}$$
a linear transformation? Justify.

   (d) Let the vector space
$$V = \{\text{functions } f(x) : \mathbb{R} \to \mathbb{R} \mid f \text{ is differentiable}\}$$
with vector addition and scalar multiplication defined pointwise as before.
Is the map $T : V \to V$ given by
$$T(f(x)) = x \cdot f(x)$$
a linear transformation? Justify.

4. True or False (justify your answers with proofs/counterexamples):

   (a) If $T : V \to W$ is a linear transformation, and $\mathbf{v}_1, \ldots \mathbf{v}_n$ are linearly independent in V, then $T(\mathbf{v}_1), \ldots T(\mathbf{v}_n)$ are linearly independent in W.

   (b) If $S, T : V \to W$ are both linear transformations that agree on a basis $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$ for V, then $S = T$.

   (c) Given $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ and $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}^2$ such that $\mathbf{v}_1 \neq \mathbf{v}_2$, there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$