
All answers must be justified appropriately.

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9. Suppose $\{\mathbf{v_1}, \mathbf{v_2}\}$ is a basis for a vector space V. Let $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ be any set of vectors in V. Prove that $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ is not a basis for V.

ANSWER:

We will show that $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ is linearly dependent. As $\{\mathbf{v_1}, \mathbf{v_2}\}$ is a basis, we can write:

$$\mathbf{w_1} = a_1\mathbf{v_1} + a_2\mathbf{v_2}$$
$$\mathbf{w_2} = b_1\mathbf{v_1} + b_2\mathbf{v_2}$$
$$\mathbf{w_3} = c_1\mathbf{v_1} + c_2\mathbf{v_2}$$

for $a_i, b_i, c_i \in \mathbb{F}$.

If $a_1 = b_1 = c_1 = \mathbf{0}$, then each of $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ is a scalar multiple of $\mathbf{v_2}$, and $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ is linearly dependent. So we may assume (possibly after reindexing $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$) that $a_1 \neq 0$.

Then

$$\mathbf{w_2} - \frac{b_1}{a_1}\mathbf{w_1} = \left(b_2 - \frac{b_1}{a_1}a_2\right)\mathbf{v_2}$$
$$\mathbf{w_3} - \frac{c_1}{a_1}\mathbf{w_1} = \left(c_2 - \frac{c_1}{a_1}a_2\right)\mathbf{v_2}$$

If $b = b_2 - \frac{b_1}{a_1}a_2$ is 0, then $\mathbf{w_1}$ and $\mathbf{w_2}$ are linearly dependent.
If $c = c_2 - \frac{c_1}{a_1}a_2$ is 0, then $\mathbf{w_1}$ and $\mathbf{w_3}$ are linearly dependent.
Therefore we may assume that $b \neq 0 \neq c$. But then the equation

$$\mathbf{v_2} = \alpha_1\mathbf{w_1} + \alpha_2\mathbf{w_2} + \alpha_3\mathbf{w_3}$$

has at least two distinct solutions:

$$(\alpha_1, \alpha_2, \alpha_3) = \left(-\frac{b_1}{a_1}b^{-1}, b^{-1}, 0\right)$$
$$(\alpha_1, \alpha_2, \alpha_3) = \left(-\frac{c_1}{a_1}c^{-1}, 0, c^{-1}\right)$$

and $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ is linearly dependent.

Therefore in all cases $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ is linearly dependent. In particular $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ is not a basis.