All answers must be justified appropriately.

From TREIL do problems:

page 115: 4.2.9
page 254: 9.1.1

(1) Of course the identity matrix $I \in \mathrm{Mat}_n(\mathbb{F})$ has the basis of eigenvectors $\mathcal{E}_n$, the standard basis of column vectors, each with eigenvalue 1. We introduced the elementary matrices as being "close to the identity." That is reflected in the fact that most elements of the basis $\mathcal{E}$ remain eigenvectors for each elementary matrix, again with eigenvalue 1.

(a) For $S_i(r)$, with $r \neq 1$, find $n-1$ elements of $\mathcal{E}$ that are eigenvectors for the eigenvalue 1. Find the remaining eigenvalue and associated eigenvector.

(b) For $X_{i,j}$ find $n-2$ elements of $\mathcal{E}$ that are eigenvectors for the eigenvalue 1. Find the remaining two eigenvalues and associated eigenvectors.

(c) For $R_{i,j}(a)$, with $a \neq 0$, find $n-1$ elements of $\mathcal{E}$ that are eigenvectors for the eigenvalue 1. Prove that this matrix cannot be diagonalized.

(2) Recall that a square matrix $A \in \mathrm{Mat}_n(\mathbb{F})$ is *nilpotent* if $A^k = 0$ for some positive integer k. Prove that a nonzero nilpotent matrix cannot be diagonalized.

(3) Let J denote the $n \times n$ matrix with *every* entry equal to 1. Prove that the vector $\mathbf{1} = (1, 1, 1, \ldots, 1)^\top$ is an eigenvector for the eigenvalue n and that any vector $\mathbf{v} = (v_1, \ldots, v_n)$ with $\sum_{i=1}^n v_i = 0$ is an eigenvector for the eigenvalue 0.

(4) Consider the $n \times n$ matrix

$$M = \begin{pmatrix} r & l & l & \ldots & l \\ l & r & l & \ldots & l \\ l & l & r & \ldots & l \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l & l & l & \ldots & r \end{pmatrix} .$$

Find the determinant of M. (You may assume $n \neq 0$ in $\mathbb{F}$.)

HINT: You can put M into echelon form.

HINT: *Alternatively:* Write M as $(r-l)I + lJ$. Then explain why a basis of eigenvectors for J (from the previous problem) is a basis of eigenvectors for M. Find the corresponding eigenvalues. The product of these eigenvalues (including multiplicities) is then the desired determinant.