I study mathematical physics, which could broadly be defined as the application of mathematics to equations and models suggested by scientific theory, with a focus on deriving rigorous results.

Much of my work has focused on the propagation of waves in a disordered background and related problems in the spectral theory of random operators. One of the most challenging problems in this area is to prove the existence of, and to study, a crossover between the "localized regime," with localized eigenfunctions and exponential suppression of propagation, and a "propagating regime," with extended eigenfunctions and wave propagation. Such a crossover is believed to occur as a function of the spectral parameter (energy or frequency) for certain random partial differential (or difference) operators generating wave motion in a disordered medium. However, the existence of the crossover has been notoriously difficult to prove.

In the mathematics literature, most papers have studied the localized side of the transition. I was involved in several of these efforts, most recently a joint paper in the Inventiones with Aizenman, Elgart, Naboko and Stolz. One of the consequences of that work is a proof that, given an operator $H$ of the type analyzed therein, if an initial wave packet $\psi_0$ is exponentially localized and has spectral support in the localized regime, then the evolved wave $\psi_t$, following the Schrödinger equation $\partial_t \psi_t = -iH\psi_t$, is uniformly exponentially localized for all time, $E(\sup_t \| e^{mX}\psi_t \|) < \infty$, where $E$ denotes disorder averaging and $|X|$ is multiplication by the length of the spatial coordinate $X$.

In contrast, almost nothing is known about the propagating regime. One exceptional case in which propagation was established and partially analyzed is a joint paper of mine, with Germinet and Klein, in the Annals. This case involves certain random Schrödinger operators in $2D$ which arise in the theory of the quantum Hall effect. Given one of these $2D$ Schrödinger operators, $H$, and a spectral parameter $\lambda \in \mathbb{R}$ in the localized regime, one may define a quantity

$$\sigma(\lambda) = \frac{1}{2\pi i} T \left( P_\lambda \left[ [P_\lambda, X], [P_\lambda, Y] \right] \right),$$

which in physical applications is the Hall conductance. Here $T$ is trace per unit volume, $[A, B] = AB - BA$, $X$ and $Y$ are multiplication by the coordinate functions, and $P_\lambda$ is the "Fermi projection" onto the spectral subspace on which $H < \lambda$. It turns out that $\sigma$ may be expressed as the index of a Fredholm operator. The stability of the index, in turn, allows one to show that $\sigma(\lambda)$ is unchanged as $\lambda$ moves through an interval in the localized regime. This fact, which has been used to explain the conductance plateaus seen in quantum Hall experiments, implies that if $\sigma$ is non-constant then there is some point $\lambda^*$ not contained in the localized regime. However, a simple perturbation argument establishes that $\sigma$ is not globally constant for an operator $H$ with weak disorder, so not all eigenstates of $H$ are localized! Based on some deeper analysis of the localized regime one can show the existence of an initial function $\psi_0$, with strong spectral support near $\lambda^*$, such that the evolution $\psi_t$ exhibits transport, i.e.,

$$\frac{1}{T} \int_0^T \| (X^2 + Y^2)^{m/2}\psi_t \| \, dt \gtrsim \text{const.} T^{\nu_m}, \text{ with } \nu_m > 0 \text{ for sufficiently large } m.$$

One focus of my current research is to study this crossover in band random matrices—random matrices with entries that vanish in a band of width $W$ around the diagonal. Here we are in the happy situation of having at our disposal explicit examples of the "extended states" regime, namely the invariant gaussian ensembles GUE, GOE and GSE. On the flip side a diagonal random matrix has independent eigenvalues and the standard basis for its eigenvectors. A crossover must occur as we interpolate between these two extremes taking the bandwidth $W$ from 1 to $N$, the matrix size. In the physics literature one finds numerical and heuristic calculations showing the crossover to occur at $W^2 \approx N$. In work in preparation, I have shown that for $W^\nu \lesssim N$, with $\nu$ an explicit exponent, the matrix has localized eigenvectors (in the standard basis) and Poisson eigenvalue correlations in a suitable large $N$ limit. It remains an intriguing question to improve the estimates to obtain $\nu = 2 - \epsilon$ and to understand what happens for $W^2 \gtrsim N$. 

Research Summary

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