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Hölder Equicontinuity of the Integrated Density of States at Weak Disorder

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Abstract. Hölder continuity, $|N_\lambda(E) - N_\lambda(E')| \leqslant C|E - E'|^\alpha$, with a constant C independent of the disorder strength λ is proved for the integrated density of states $N_\lambda(E)$ associated to a discrete random operator $H = H_o + \lambda V$ consisting of a translation invariant hopping matrix H_o and i.i.d. single site potentials V with an absolutely continuous distribution, under a regularity assumption for the hopping term.

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1. Introduction

Random operators on $\ell^2(\mathbb{Z}^d)$ of the general form

$$H_\omega = H_o + \lambda V_\omega, \tag{1}$$

play a central role in the theory of disordered materials, where:

(1) $V_\omega \psi(x) = \omega(x)\psi(x)$ with $\omega(x)$, $x \in \mathbb{Z}^d$, independent identically distributed random variables whose common distribution is $\rho(\omega)\mathrm{d}\omega$ with ρ a bounded function. The coupling $\lambda \in \mathbb{R}$ is called the *disorder strength*.
(2) H_o is a bounded translation invariant operator, i.e., $\left[S_\xi, H_o\right] = 0$ for each translation $S_\xi \psi(x) = \psi(x - \xi)$, $\xi \in \mathbb{Z}^d$.

The *density of states measure* for an operator H_ω of the form Equation (1) is the (unique) Borel measure $\mathrm{d}N_\lambda(E)$ on the real line defined by

$$\int f(E)\mathrm{d}N_\lambda(E) = \lim_{L\to\infty} \frac{1}{\#\left\{x \in \mathbb{Z}^d : |x| < L\right\}} \sum_{x:|x|<L} \langle \delta_x, f(H_\omega)\delta_x \rangle,$$

and the *integrated density of states* $N_\lambda(E)$ is

$$N_\lambda(E) := \int_{(-\infty,E)} \mathrm{d}N_\lambda(\varepsilon).$$

It is a well known consequence, e.g., reference [6], of the translation invariance of the distribution of H_ω that the density of states exists and equals

$$N_\lambda(E) = \int_\Omega \langle \delta_0, P_{(-\infty,E)}(H_\omega)\delta_0\rangle \mathrm{d}\mathbb{P}(\omega), \qquad \text{every } E \in \mathbb{R};$$

for $\mathbb{P}$ almost every ω, where $\mathbb{P}$ is the joint probability distribution for ω and $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ is the probability space.

The density of states measure is an object of fundamental physical interest. For example, the free energy f per unit volume of a system of non-interacting identical Fermions, each governed by a Hamiltonian H_ω of the form Equation (1), is

$$f(\mu, \beta) = -\beta \int \ln(1 + \mathrm{e}^{-\beta(E-\mu)})\mathrm{d}N_\lambda(E),$$

where β is the inverse temperature and μ is the chemical potential. Certain other thermodynamic quantities (density, heat capacity, etc.) of the system can also be expressed in terms of N_λ.

Our main result is equicontinuity of the family $\{N_\lambda(\cdot), \lambda > 0\}$ within a class of Hölder continuous functions, that is

$$N_\lambda(E+\delta) - N_\lambda(E-\delta) \leqslant C_\alpha\, \delta^\alpha, \qquad \text{for all } \lambda > 0, \tag{2}$$

under appropriate hypotheses on H_o. The exponent $\alpha < 1$ depends on H_o as well as the probability density, with $\alpha = \frac{1}{2}$ at generic E for a large class of hopping terms if ρ is compactly supported.

A bound of the form Equation (2) for the integrated density of states associated to a continuum random Schrödinger operator is implicit in Theorem 1.1 of reference [1], although uniformity in λ is not explicitly noted there. The tools of reference [1] carry over easily to the discrete context to give an alternative proof of Equation (2). However the methods employed herein are in fact quite different from those of reference [1], and may be interesting in and of themselves.

The main point of Equation (2) is the uniformity of the bound as $\lambda \to 0$, since the well known *Wegner estimate* [9], see also [7, Theorem 8.2],

$$\frac{\mathrm{d}N_\lambda(E)}{\mathrm{d}E} \leqslant \frac{\|\rho\|_\infty}{\lambda}, \tag{3}$$

implies that $N_\lambda(E)$ is in fact Lipschitz continuous,

$$N_\lambda(E+\delta) - N_\lambda(E-\delta) \leqslant \frac{\|\rho\|_\infty}{\lambda}\, 2\delta. \tag{4}$$

However, the Lipschitz constant $\|\rho\|_\infty/\lambda$ in Equation (4) diverges as $\lambda \to 0$. Such a singularity is inevitable for a bound which makes no reference to the hopping term, since $\mathrm{d}N_\lambda(E) = \lambda^{-1}\rho(E/\lambda)\mathrm{d}E$ for $H_o = 0$, as may easily be verified. However if the background itself has an absolutely continuous density of states, the Wegner estimate is far from optimal at weak disorder.

The translation invariant operator H_o may be written as a superposition of translations,

$$H_o = \sum_{\xi} \check{\varepsilon}(\xi) S_\xi ,$$

where

$$\check{\varepsilon}(\xi) = \int_{T^d} \varepsilon(\mathbf{q}) \mathrm{e}^{-\mathrm{i}\xi\cdot\mathbf{q}} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d},$$

is the inverse Fourier transform of a bounded real function ε on the torus $T^d = [0, 2\pi)^d$, called the symbol of H_o. For any bounded measurable function f,

$$f(H_o) = \sum_{\xi\in\mathbb{Z}^d} \left[\int_{T^d} f(\varepsilon(\mathbf{q})) \mathrm{e}^{-\mathrm{i}\xi\cdot\mathbf{q}} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \right] S_\xi ,$$

from which it follows that the density of states $N_o(E)$ for H_o obeys

$$\int f(E) \mathrm{d}N_o(E) = \int_{T^d} f(\varepsilon(\mathbf{q})) \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}.$$

In particular,

$$N_o(E) = \int_{\{\varepsilon(\mathbf{q})<E\}} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}.$$

We define a *regular point* for ε to be a point $E \in \mathbb{R}$ at which

$$N_o(E+\delta) - N_o(E-\delta) \leqslant \Gamma(E)\,\delta, \tag{5}$$

for some $\Gamma(E) < \infty$. In particular if ε is C^1 and $\nabla\varepsilon$ is non-zero on the level set $\{\varepsilon(\mathbf{q}) = E\}$, then E is a regular point. For example, with H_o the discrete Laplacian on $\ell^2(\mathbb{Z})$,

$$H_o\psi(x) = \psi(x+1) + \psi(x-1),$$

we have the symbol $\varepsilon(q) = 2\cos(q)$ and every $E \in (-2, 2)$ is a regular point. However at the band edges, $E = \pm 2$, the difference on the left hand side of Equation (5) is only $\mathcal{O}(\delta^{\frac{1}{2}})$, and these points are not regular points. We consider the behavior of $N_\lambda(E)$ at such 'points of order α', here $\alpha = 1/2$, in Theorem 3 below.

Our main result involves the density of states of H_λ at a regular point:

THEOREM 1. *Suppose $\int |\omega|^q \rho(\omega)\mathrm{d}\omega < \infty$ for some $2 < q < \infty$ or that ρ is compactly supported, in which case set $q = \infty$. If E is a regular point for ε, then there is $C_q = C_q(\rho, \Gamma(E)) < \infty$ such that*

$$N_\lambda(E+\delta) - N_\lambda(E-\delta) \leqslant \Gamma(E)\,\delta + C_q\, \lambda^{\frac{1}{3}(1+\frac{2}{q})} \delta^{\frac{1}{3}(1-\frac{2}{q})} \tag{6}$$

for all $\lambda, \delta \geqslant 0$.

For very small δ, namely

$$\frac{\delta}{\lambda} \lesssim \lambda^{\frac{1}{3}(1+\frac{2}{q})} \delta^{\frac{1}{3}(1-\frac{2}{q})},$$

the Wegner bound Equation (3) is stronger than Equation (6).[1] Thus Theorem 1 is useful only for

$$\delta \gtrsim \lambda^{\frac{2q+1}{q+1}}.$$

Combining the Wegner estimate and Theorem 1 for these separate regions yields the following:

COROLLARY 2. *Under the hypotheses of Theorem* 1, *there is* $C_q < \infty$, *with* $C_q = C_q(\rho, \Gamma(E))$, *such that*

$$N_\lambda(E+\delta) - N_\lambda(E-\delta) \leqslant C_q \delta^{\frac{1}{2}(1-\frac{1}{2q+1})} \tag{7}$$

for all $\lambda, \delta \geqslant 0$.

Thus, the integrated density of states is Hölder equi-continuous of order $\frac{1}{2}$ as $\lambda \to 0$ (if ρ is compactly supported).

The starting point for our analysis of the density of states is a well-known formula relating $\mathrm{d}N_\lambda$ to the resolvent of H_ω,

$$\frac{\mathrm{d}N_\lambda(E)}{\mathrm{d}E} = \lim_{\eta \downarrow 0} \frac{1}{\pi} \int_\Omega \mathrm{Im} \langle \delta_0, (H_\omega - E - \mathrm{i}\eta)^{-1} \delta_0 \rangle \mathrm{d}\mathbb{P}(\omega).$$

The general idea of the proof is to express $\mathrm{Im}\langle \delta_0, (H_\omega - E - \mathrm{i}\eta)^{-1}\delta_0\rangle$ using a finite resolvent expansion to second order

$$\begin{aligned}(H_\omega - E - \mathrm{i}\eta)^{-1} \\ &= (H_o - E - \mathrm{i}\eta)^{-1} - \lambda (H_o - E - \mathrm{i}\eta)^{-1} V_\omega (H_o - E - \mathrm{i}\eta)^{-1} + \\ &\quad + \lambda^2 (H_o - E - \mathrm{i}\eta)^{-1} V_\omega (H_\omega - E - \mathrm{i}\eta)^{-1} V_\omega (H_o - E - \mathrm{i}\eta)^{-1},\end{aligned} \tag{8}$$

and to use the Wegner bound Equation (3) to estimate the last term, with the resulting factor of $1/\lambda$ controlled by the factor λ^2.

Here is a simplified version of the argument which works if E falls outside the spectrum of H_o and $\psi_E = (H_o - E)^{-1}\delta_0 \in \ell^1(\mathbb{Z}^d)$. The first two terms of Equation (8) are bounded and self-adjoint when $\eta = 0$, so

$$\begin{aligned}&\lim_{\eta \downarrow 0} \frac{1}{\pi} \int_\Omega \mathrm{Im}\langle \delta_0, (H_\omega - E - \mathrm{i}\eta)^{-1} \delta_0 \rangle \mathrm{d}\mathbb{P}(\omega) \\ &= \lambda^2 \lim_{\eta \downarrow 0} \frac{1}{\pi} \int_\Omega \mathrm{Im}\langle \psi E, V\omega (H_\omega - E - \mathrm{i}\eta)^{-1} V_\omega \psi_E \rangle \mathrm{d}\mathbb{P}(\omega) \\ &\leqslant \lambda^2 \lim_{\eta \downarrow 0} \sum_{x,y} |\psi_E(x)||\psi_E(y)| \times \\ &\quad \times \frac{\eta}{\pi} \int_\Omega \left| \omega(x)\omega(y) \left\langle \delta_x, ((H_\omega - E)^2 + \eta^2)^{-1} \delta_y \right\rangle \right| \mathrm{d}\mathbb{P}(\omega).\end{aligned}$$

[1]We thank M. Disertori for this observation.

If ρ is, say, compactly supported, then

$$\lim_{\eta\downarrow 0}\frac{\eta}{\pi}\int_\Omega\left|\omega(x)\omega(y)\left\langle\delta_x,\left((H_\omega-E)^2+\eta^2\right)^{-1}\delta_y\right\rangle\right|\mathrm{d}\mathbb{P}(\omega)$$
$$\lesssim\lim_{\eta\downarrow 0}\frac{\eta}{\pi}\int_\Omega\left\langle\delta_x,\left((H_\omega-E)^2+\eta^2\right)^{-1}\delta_y\right\rangle\mathrm{d}\mathbb{P}(\omega)\lesssim\frac{1}{\lambda},$$

by the Wegner bound, and therefore

$$\frac{\mathrm{d}N_\lambda(E)}{\mathrm{d}E}\lesssim\lambda\|\psi_E\|_1^2,\qquad\text{for } E\notin\sigma(H_o).\tag{9}$$

We have used second order perturbation theory to 'boot-strap' the Wegner estimate and obtain an estimate of lower order in λ. Unfortunately, as ρ was assumed compactly supported, E is not in the spectrum of H_λ for sufficiently small λ, and thus $\mathrm{d}N_\lambda(E)/\mathrm{d}E=0$. So, in practice, Equation (9) is not a useful bound.

Nonetheless, in the cases covered by Theorem 1, H_λ can have spectrum in a neighborhood of E, even for small λ, since E may be in the interior of the spectrum of H_o. Although, the above argument does not go through, we shall exploit the translation invariance of the distribution of H_ω by introducing a Fourier transform on the Hilbert space of 'random wave functions', complex valued functions $\Psi(x,\omega)$ of $(x,\omega)\in\ell^2(\mathbb{Z}^d)\times\Omega$ with

$$\sum_x\int_\Omega|\Psi(x,\omega)|^2\mathrm{d}\mathbb{P}(\omega)<\infty.$$

Under this Fourier transform an integral $\int_\Omega$ of a matrix element of $f(H_\omega)$ is replaced by an integral $\int_{T^d}$ over the d-torus of a matrix element of $f(\widehat{H}_{\mathbf{k}})$, with $\widehat{H}_{\mathbf{k}}$ a certain family of operators on $L^2(\Omega)$ (see Equation (16)). Off the set $S_\varepsilon:=\{\mathbf{k}\in T^d||\varepsilon(\mathbf{k})-E|>\epsilon\}$ with $\epsilon\gg\delta$, we are able to carry out an argument similar to that which led to Equation (9). To prove Theorem 1, we shall directly estimate

$$N(E+\delta)-N(E-\delta)=\int_\Omega\langle\delta_0,P_\delta(H_\omega)\delta_0\rangle\mathrm{d}\mathbb{P}(\omega),$$

with P_δ the characteristic function of the interval $[E-\delta,E+\delta]$, because the integrand on the r.h.s. is bounded by 1. Since E is a regular point, the error in restricting to S_ε will be bounded by $\Gamma(E)\varepsilon$. Choosing ε optimally will lead to Theorem 1.

More generally, we say that E is a point of order α for ε, if there exists $\Gamma(E;\alpha)$ such that

$$N_o(E+\delta)-N_o(E-\delta)\leqslant\Gamma(E;\alpha)\delta^\alpha.$$

If $E\notin\sigma(H_o)$, we say that E is a point of order ∞ and set $\Gamma(E;\infty)=0$. For points of order α, we have the following extension of Theorem 1.

THEOREM 3. *Suppose* $\int |\omega|^q \rho(\omega)\mathrm{d}\omega < \infty$ *for some* $2 < q < \infty$ *or that* ρ *is compactly supported, in which case set* $q = \infty$. *If* E *is a point of order* $\alpha \leqslant \infty$ *for* ε, *then there is* $C_{q,\alpha} = C_{q,\alpha}(\rho, \Gamma(E;\alpha)) < \infty$ *such that*

$$N_\lambda(E+\delta) - N_\lambda(E-\delta) \leqslant \Gamma(E;\alpha)\delta^\alpha + C_{q,\alpha}\left[\lambda^{1+\frac{2}{q}}\delta^{1-\frac{2}{q}}\right]^{\frac{1}{1+\frac{2}{\alpha}}} \tag{10}$$

for all $\lambda, \delta \geqslant 0$.

When $\alpha = \infty$ and $q = \infty$, so $E \notin \sigma(H_o)$ and ρ is compactly supported, the result is technically true but uninteresting since $E \notin \sigma(H_\lambda)$ for small λ, as discussed above. However for $q < \infty$, we need not have that ρ is compactly supported, and $E \notin \sigma(H_o)$ may still be in the spectrum of H_λ for arbitrarily small λ. In this case, Equation (10) signifies that

$$N_\lambda(E+\delta) - N_\lambda(E-\delta) \leqslant C_{q,\infty}\lambda^{1+\frac{2}{q}}\delta^{1-\frac{2}{q}},$$

which in fact improves on the Wegner bound for appropriate λ, δ.

As above, we may use the Wegner bound for δ very small to improve on Equation (10):

COROLLARY 4. *Under the hypotheses of Theorem* 3, *there is* $C_{q,\alpha} = C_{q,\alpha}(\rho, \Gamma(E;\alpha)) < \infty$ *such that*

$$N_\lambda(E+\delta) - N_\lambda(E-\delta) \leqslant C_{q;\alpha}\delta^{\frac{\alpha}{\alpha+1}\left(1-\frac{1}{\frac{\alpha+1}{\alpha}q+1}\right)}$$

for all $\lambda, \delta \geqslant 0$.

The inspiration for these results is the (non-rigorous) renormalized perturbation theory for $\mathrm{d}N_\lambda$ which has appeared in the physics literature, e.g., reference [8] and references therein. If $\int \omega\rho(\omega)\mathrm{d}\omega = 0$ and $\int \omega^2\rho(\omega)\mathrm{d}\omega = 1$, as can always be achieved by shifting the origin of energy and re-scaling λ, then the central result of that analysis is that

$$\frac{\mathrm{d}N_\lambda(E)}{\mathrm{d}E} \approx \frac{1}{\pi}\mathrm{Im}\left\langle\delta_0\left(H_o - E - \lambda^2\Gamma_\lambda(E)\right)^{-1}\delta_0\right\rangle,$$

where $\Gamma_\lambda(E)$, the so-called 'self energy', satisfies $\mathrm{Im}\,\Gamma_\lambda(E) > 0$ with

$$\lim_{\lambda\to 0}\mathrm{Im}\,\Gamma_\lambda(E) \approx \lim_{\eta\to 0}\mathrm{Im}\langle\delta_0, (H_o - E - \mathrm{i}\eta)^{-1}\delta_0\rangle = \pi\frac{\mathrm{d}N_o(E)}{\mathrm{d}E}.$$

Up to a point, the self-energy analysis may be followed rigorously. Specifically, one can show (see Section 2):

PROPOSITION 1.1. *If* $\int \omega\rho(\omega)\mathrm{d}\omega = 0$ *and* $\int \omega^2\rho(\omega)\mathrm{d}\omega = 1$, *then for each* $\lambda > 0$ *there is a map* Γ_λ *from* $\{\mathrm{Im}\, z > 0\}$ *to the translation invariant operators with non-negative imaginary part on* $\ell^2(\mathbb{Z}^2)$ *such that*

$$\int_\Omega (H_\omega - z)^{-1}\mathrm{d}\mathbb{P}(\omega) = \left(H_o - z - \lambda^2\Gamma_\lambda(z)\right)^{-1}, \tag{11}$$

and for fixed $z \in \{\mathrm{Im}\, z > 0\}$

$$\lim_{\lambda\to 0}\langle\delta_x, \Gamma_\lambda(z)\delta_y\rangle = \langle\delta_0, (H_o - z)^{-1}\delta_0\rangle\,\delta_{x,y}. \tag{12}$$

However there is *a priori* no uniformity in z for the convergence in Equation (12), so for fixed λ we may conclude nothing about

$$\lim_{\eta\downarrow 0}\left(H_o - E - \mathrm{i}\eta - \lambda^2\Gamma_\lambda(E + \mathrm{i}\eta)\right)^{-1}.$$

Still, one is left feeling that Theorem 1 and Corollary 2 are not optimal, and the 'standard wisdom' is that something like the following is true.

CONJECTURE 5. Let ρ have moments of all orders, i.e., $\int |\omega|^q\rho(\omega) < \infty$ for all $q \geqslant 1$. Given $E_o \in \mathbb{R}$, if there is $\delta > 0$ such that on the set $\{\mathbf{q} : |\varepsilon(\mathbf{q}) - E_o| < \delta\}$ the symbol ε is C^1 with $\nabla\varepsilon(\mathbf{q}) \neq 0$, then there is $C_\delta < \infty$ such that

$$\frac{\mathrm{d}N_\lambda(E)}{\mathrm{d}E} \leqslant C_\delta$$

for all $\lambda \in \mathbb{R}$ and $E \in [E_o - \frac{1}{2}\delta, E_o + \frac{1}{2}\delta]$.

Remark. The requirement that ρ have moments of all orders is simply the minimal requirement for the infinite perturbation series for $(H_o - z - \lambda V_\omega)^{-1}$ to have finite expectation at each order (for $\mathrm{Im}\, z > 0$). In fact, this may be superfluous, as suggested by the example of Cauchy randomness, for which the density of states can be explicitly computed, see reference [7]:

$$\mathrm{d}N_\lambda(E) = \frac{1}{\pi}\int_{T^d}\frac{\lambda}{(\varepsilon(\mathbf{q}) - E)^2 + \lambda^2}\frac{\mathrm{d}\mathbf{q}}{(2\pi)^d}, \quad \text{for } \rho(\omega) = \frac{1}{\pi}\frac{1}{1+\omega^2},$$

although $\int \rho(\omega)|\omega|^q = \infty$ for every $q \geqslant 1$.

2. Translation Invariance, Augmented Space, and a Fourier Transform

The joint probability measure $\mathbb{P}(\omega)$ for the random function $\omega: \mathbb{Z}^d \to \mathbb{R}$ is

$$\mathrm{d}\mathbb{P}(\omega) := \prod_{x\in\mathbb{Z}^d}\rho(\omega(x))\mathrm{d}\omega(x)$$

on the probability space $\Omega=\mathbb{R}^{\mathbb{Z}^d}$. Clearly, $\mathbb{P}(\omega)$ is invariant under the translations $\tau_\xi\colon \Omega\to\Omega$ defined by

$$\tau_\xi\omega(x)=\omega(x-\xi).$$

In particular, since

$$S_\xi H_\omega S_\xi^\dagger = H_o + V_{\tau_\xi\omega} = H_{\tau_\xi\omega}, \tag{13}$$

H_ω and $S_\xi H_\omega S_\xi^\dagger$ are identically distributed for any $\xi\in\mathbb{Z}^d$.

To express this invariance in operator theoretic terms, we introduce the fibred action of H_ω on the Hilbert space $L^2(\Omega;\ell^2(\mathbb{Z}^d))$ – the space of 'random wave functions' – namely,

$$\Psi(\omega)\mapsto H_\omega\Psi(\omega).$$

We identify $L^2(\Omega;\ell^2(\mathbb{Z}^d))$ with $L^2(\Omega\times\mathbb{Z}^d)$ and denote the action of H_ω on the latter space by $\mathbf{H}$, so

$$[\mathbf{H}\Psi](\omega,x)=\sum_\xi \check{\varepsilon}(\xi)\Psi(\omega,x-\xi)+\lambda\omega(x)\Psi(\omega,x).$$

The following elementary identity relates $\int_\Omega f(H_\omega)\mathrm{d}\mathbb{P}(\omega)$ to $f(\mathbf{H})$, for any bounded measurable function f,

$$\int_\Omega \mathrm{d}\mathbb{P}(\omega)\langle\delta_x, f(H_\omega)\delta_y\rangle = \langle\mathbb{E}^\dagger\delta_x, f(\mathbf{H})\mathbb{E}^\dagger\delta_y\rangle, \tag{14}$$

where $\mathbb{E}^\dagger$ is the adjoint of the linear expectation map $\mathbb{E}\colon L^2(\Omega\times\mathbb{Z}^d)\to\ell^2(\mathbb{Z}^d)$ defined by

$$[\mathbb{E}\Psi](x)=\int_\Omega \Psi(\omega,x)\mathrm{d}\mathbb{P}(\omega).$$

Note that $\mathbb{E}^\dagger$ is an isometry from $\ell^2(\mathbb{Z}^d)$ onto the subspace of functions independent of ω – 'non-random functions'.

The general fact that averages of certain quantities depending on H_ω can be represented as matrix elements of $\mathbf{H}$ is known, and is sometimes called the 'augmented space representation' (e.g., references [3–5]) where 'augmented space' refers to the Hilbert space $L^2(\Omega\times\mathbb{Z}^d)$. There are 'augmented space' formulae other than Equation (14), such as

$$\int_\Omega \mathrm{d}\mathbb{P}(\omega)\omega(x)\omega(y)\langle\delta_x, f(H_\omega)\delta_y\rangle = \langle\mathbb{E}^\dagger\delta_x, \mathbf{V}f(\mathbf{H})\mathbf{V}\mathbb{E}^\dagger\delta_y\rangle, \tag{15}$$

with $\mathbf{V}$ defined below, and

$$\int_\Omega \mathrm{d}\mathbb{P}(\omega)\langle\delta_x, f(H_\omega)\delta_0\rangle\langle\delta_0, g(H_\omega)\delta_y\rangle \;=\; \langle\mathbb{E}^\dagger\delta_x, f(\mathbf{H})P_0 g(\mathbf{H})\mathbb{E}^\dagger\delta_y\rangle,$$

where P_0 denotes the projection $P_0\Psi(\omega,x)=\Psi(\omega,0)$ if $x=0$ and 0 otherwise. The first of these (Equation (15)) will play a roll in the proof of Theorem 1.

There are two natural groups of unitary translations on $L^2(\Omega\times\mathbb{Z}^d)$:

$$S_\xi \Psi(\omega, x) = \Psi(\omega, x - \xi),$$

and

$$T_\xi \Psi(\omega, x) = \Psi(\tau_{-\xi}\omega, x).$$

Note that these groups commute: $\left[S_\xi, T_{\xi'}\right] = 0$ for every $\xi, \xi' \in \mathbb{Z}^d$. A key observation is that the distributional invariance of H_ω, Equation (13), results in the *invariance* of $\mathbf{H}$ under the combined translations $T_\xi S_\xi = S_\xi T_\xi$:

$$S_\xi T_\xi \mathbf{H} T_\xi^\dagger S_\xi^\dagger = \mathbf{H}.$$

In fact, let us define

$$\mathbf{H}_o = \sum_\xi \check{\varepsilon}(\xi) S_\xi, \quad \mathbf{V}\Psi(\omega, x) = \omega(x)\Psi(\omega, x).$$

Then $\mathbf{H} = \mathbf{H}_o + \lambda \mathbf{V}$ where $\mathbf{H}_o$ commutes with S_ξ and T_ξ while for $\mathbf{V}$ we have

$$\mathbf{V} S_\xi = T_{-\xi} \mathbf{V}.$$

To exploit this translation invariance of $\mathbf{H}$, we define a Fourier transform which diagonalizes the translations $S_\xi T_\xi$ (and therefore partially diagonalizes $\mathbf{H}$). The result is a unitary map $\mathcal{F}$: $L^2(\Omega \times \mathbb{Z}^d) \to L^2(\Omega \times T^d)$, with T^d the d-torus $[0, 2\pi)^d$. Let us define $\mathcal{F}$ first on functions having finite support in $\mathbb{Z}^d$ by

$$\mathcal{F}\Psi(\omega, \mathbf{k}) = \sum_\xi \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\xi} \Psi(\tau_{-\xi}\omega, -\xi).$$

It is easy to verify, using well-known properties of the usual Fourier series mapping $\ell^2(\mathbb{Z}^d) \to L^2(T^d)$, that $\mathcal{F}$ extends to a unitary isomorphism $L^2(\Omega \times \mathbb{Z}^d) \to L^2(\Omega \times T^d)$, i.e. that $\mathcal{F}\mathcal{F}^\dagger = 1$ and $\mathcal{F}^\dagger\mathcal{F} = 1$ where $\mathcal{F}^\dagger$ is the adjoint map

$$\mathcal{F}^\dagger \widehat{\Psi}(\omega, x) = \int_{T^d} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot x} \widehat{\Psi}(\tau_{-x}\omega, \mathbf{k}) \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d}.$$

Another way of looking at $\mathcal{F}$ is to define for each $\mathbf{k} \in T^d$ an operator $\mathcal{F}_\mathbf{k}$: $L^2(\Omega \times \mathbb{Z}^d) \to L^2(\Omega)$ by

$$\mathcal{F}_\mathbf{k}\Psi = \lim_{L\to\infty} \sum_{|\xi|<L} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\xi} \mathcal{J} S_\xi T_\xi \Psi,$$

where $\mathcal{J}$ is the evaluation map $\mathcal{J}\Psi(\omega) = \Psi(\omega, 0)$. The maps $\mathcal{F}_\mathbf{k}$ are *not* bounded, but are densely defined with $\mathcal{F}_\mathbf{k}\Psi \in L^2(\Omega)$ for almost every $\mathbf{k}$, and

$$\mathcal{F}\Psi(\omega, \mathbf{k}) = \mathcal{F}_\mathbf{k}\Psi(\omega) \quad \text{a.e. } \omega, \mathbf{k}.$$

If we look at $L^2(\Omega \times T^d)$ as the direct integral $\int^\oplus \mathrm{d}\mathbf{k} L^2(\Omega)$, then

$$\mathcal{F} = \int^\oplus \mathrm{d}\mathbf{k} \mathcal{F}_\mathbf{k}.$$

This Fourier transform diagonalizes the combined translation $S_\xi T_\xi$,

$$\mathcal{F}_{\mathbf{k}} S_\xi T_\xi = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\xi} \mathcal{F}_{\mathbf{k}},$$

as follows from the following identities for S and T,

$$\mathcal{F}_{\mathbf{k}} T_\xi = T_\xi \mathcal{F}_{\mathbf{k}}, \quad \mathcal{F}_{\mathbf{k}} S_\xi = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\xi} T_{-\xi} \mathcal{F}_{\mathbf{k}},$$

where, on the right hand sides, T_ξ denotes the operator $T_\xi \psi(\omega) = \psi(\tau_{-\xi}\omega)$ on $L^2(\Omega)$. Furthermore, explicit computation shows that

$$\mathcal{F}_{\mathbf{k}} \mathbf{V} = \omega(0) \mathcal{F}_{\mathbf{k}},$$

where $\omega(0)$ denotes the operator of multiplication by the random variable $\omega(0)$, $\psi(\omega) \mapsto \omega(0)\psi(\omega)$. Putting this all together yields:

PROPOSITION 2.1. *Under the natural identification of $L^2(\Omega, T^d)$ with the direct integral $\int^{\oplus} \mathrm{d}\mathbf{k} L^2(\Omega)$, the operator $\widehat{\mathbf{H}} = \mathcal{F}\mathbf{H}\mathcal{F}^\dagger$ is partially diagonalized, $\widehat{\mathbf{H}} = \int^{\oplus} \widehat{H}_{\mathbf{k}}$, with $\widehat{H}_{\mathbf{k}}$ operators on $L^2(\Omega)$ given by the following formula*

$$\widehat{H}_{\mathbf{k}} = \sum_\xi \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\xi} \check{\varepsilon}(-\xi) T_\xi + \lambda\omega(0).$$

Let us introduce for each $\mathbf{k} \in T^d$,

$$\widehat{H}^o_{\mathbf{k}} := \sum_\xi \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\xi} \check{\varepsilon}(-\xi) T_\xi = \sum_\xi \left[\int_{T^d} \varepsilon(\mathbf{q}+\mathbf{k}) \mathrm{e}^{\mathrm{i}\xi\cdot\mathbf{q}} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^d} \right] T_\xi,$$

so $\widehat{H}_{\mathbf{k}} = \widehat{H}^o_{\mathbf{k}} + \lambda\omega(0)$. Note that

$$\widehat{H}^o_{\mathbf{k}} \chi_\Omega = \varepsilon(\mathbf{k}) \chi_\Omega,$$

where $\chi_\Omega(\omega) = 1$ for every $\omega \in \Omega$. That is, χ_Ω is an eigenvector for $H^o_{\mathbf{k}}$.[2]

Applying the Fourier transform $\mathcal{F}$ to the right hand side of the "augmented space" formula Equation (14) we obtain the following beautiful identity, central to this work:

$$\int_\Omega \mathrm{d}\mathbb{P}(\omega) \langle \delta_x, f(H_\omega)\delta_y \rangle = \int_{T^d} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(x-y)} \langle \chi_\Omega, f(\widehat{H}_{\mathbf{k}})\chi_\Omega \rangle. \tag{16}$$

Similarly, we obtain

$$\begin{aligned} &\int_\Omega \mathrm{d}\mathbb{P}(\omega)\omega(x)\omega(y) \langle \delta_x, f(H_\omega)\delta_y \rangle \\ &\quad = \int_{T^d} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(x-y)} \langle \omega(0)\chi_\Omega, f(\widehat{H}_{\mathbf{k}})\omega(0)\chi_\Omega \rangle \end{aligned} \tag{17}$$

[2]In fact, if ε is almost everywhere non-constant (so H_o has no eigenvalues) then $\varepsilon(k)$ is the *unique* eigenvalue for $\widehat{H}^o_{\mathbf{k}}$ and the remaining spectrum of $\widehat{H}^o_{\mathbf{k}}$ is infinitely degenerate absolutely continuous spectrum. One way to see this is to let $\phi_n(v)$ be the orthonormal polynomials with respect to the weight $\rho(v)$, and look at the action of $\widehat{H}^o_{\mathbf{k}}$ on the basis for $L^2(\Omega)$ consisting of products of the form $\prod_{x\in\mathbb{Z}^d} \phi_{n(x)}(\omega(x))$ with only finitely many $n(x) \neq 0$.

from Equation (15). Related formulae have been used, for example, to derive the Aubry duality between strong and weak disorder for the almost Mathieu equation, see reference [2] and references therein.

As a first application of Equation (16), let us prove the existence of the self energy (Proposition 1.1) starting from the identity

$$\int_{\Omega} \mathrm{d}\mathbb{P}(\omega) \left\langle \delta_0, (H_\omega - z)^{-1} \delta_0 \right\rangle = \int_{T^d} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} \left\langle \chi_\Omega, \left(\widehat{H}_{\mathbf{k}} - z\right)^{-1} \chi_\Omega \right\rangle.$$

Proof of Proposition 1.1. Since χ_Ω is an eigenvector of $\widehat{H}^o_{\mathbf{k}}$ and

$$\langle \chi_\Omega, \omega(0)\chi_\Omega \rangle = \int \omega\rho(\omega)\mathrm{d}\omega = 0,$$

the Feschbach mapping implies

$$\left\langle \chi_\Omega, \left(\widehat{H}_{\mathbf{k}} - z\right)^{-1} \chi_\Omega \right\rangle \;=\; \left(\varepsilon(k) - z - \lambda^2 \Gamma_\lambda(z; \mathbf{k})\right)^{-1}, \tag{18}$$

with

$$\Gamma_\lambda(z; \mathbf{k}) = \left\langle \omega(0)\chi_\Omega, \left(P^\perp \widehat{H}_{\mathbf{k}} P^\perp - z\right)^{-1} \omega(0)\chi_\Omega \right\rangle,$$

where $P^\perp$ denotes the projection onto the orthogonal complement of χ_Ω in $L^2(\Omega)$.

Let the self energy $\Gamma_\lambda(z)$ be the translation invariant operator with symbol $\Gamma_\lambda(z; \mathbf{k})$, i.e.,

$$\langle \delta_x, \Gamma_\lambda(z)\delta_y \rangle = \int_{T^d} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(x-y)} \Gamma_\lambda(z; \mathbf{k}) \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d}.$$

Clearly $\Gamma_\lambda(z)$ is bounded with non-negative imaginary part. Furthermore by Equations (16) and (18), the identity Equation (11) holds, namely

$$\int_{\Omega} (H_\omega - z)^{-1} \mathrm{d}\mathbb{P}(\omega) = \left(H_o - z - \lambda^2 \Gamma_\lambda(z)\right)^{-1}.$$

It is clear that

$$\lim_{\lambda \to 0} \Gamma_\lambda(z; \mathbf{k}) = \left\langle \omega(0)\chi_\Omega, \left(\widehat{H}^o_{\mathbf{k}} - z\right)^{-1} \omega(0)\chi_\Omega \right\rangle,$$

from which Equation (12) follows easily. □

3. Proofs

We first prove Theorem 1 and then describe modifications of the proof which imply Theorem 3.

3.1. PROOF OF THEOREM 1

Fix a regular point E for ε, and for each $\delta > 0$ let

$$f_\delta(t) = \frac{1}{2}\left(\chi_{(E-\delta,E+\delta)}(t) + \chi_{[E-\delta,E+\delta]}(t)\right) = \begin{cases} 1 & t, \in (E-\delta, E+\delta), \\ \frac{1}{2} & t = E \pm \delta, \\ 0 & t \notin [E-\delta, E+\delta]. \end{cases}$$

Since $N_\lambda(E)$ is continuous (see Equation (4)),

$$N_\lambda(E+\delta) - N_\lambda(E-\delta) = \int_\Omega \langle \delta_0, f_\delta(H_\omega)\delta_0\rangle \mathrm{d}\mathbb{P}(\omega).$$

Thus, in light of Equation (16), our task is to show that

$$\int_{T^d} \langle \chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}})\chi_\Omega\rangle \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} \leqslant \Gamma(E)\,\delta + C_q\, \lambda^{\frac{1}{3}\left(1+\frac{2}{q}\right)} \delta^{\frac{1}{3}\left(1-\frac{2}{q}\right)}, \tag{19}$$

with a constant C_q independent of δ and λ. Note that for each $\mathbf{k} \in T^d$

$$|\langle \chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}})\chi_\Omega\rangle| \leqslant 1,$$

so we can afford to neglect a set of Lebesgue measure $\lambda^{\frac{1}{3}\left(1+\frac{2}{q}\right)} \delta^{\frac{1}{3}\left(1-\frac{2}{q}\right)}$ on the left-hand side of Equation (19).

Consider $\mathbf{k} \in T^d$ with $|\varepsilon(\mathbf{k}) - E| > \delta$. Then

$$f_\delta(\widehat{H}^o_{\mathbf{k}})\chi_\Omega = f_\delta(\varepsilon(\mathbf{k}))\chi_\Omega = 0.$$

Thus

$$\begin{aligned} \langle \chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}})\chi_\Omega\rangle &= \left\langle \chi_\Omega, \left(f_\delta(\widehat{H}_{\mathbf{k}}) - f_\delta(\widehat{H}^o_{\mathbf{k}})\right)\chi_\Omega\right\rangle \\ &= \lim_{\eta\to 0} \frac{1}{\pi}\int_{E-\delta}^{E+\delta} \mathrm{Im}\left\langle \chi_\Omega, \left(\frac{1}{\widehat{H}_{\mathbf{k}} - t - \mathrm{i}\eta} - \frac{1}{\widehat{H}^o_{\mathbf{k}} - t - \mathrm{i}\eta}\right)\chi_\Omega\right\rangle \mathrm{d}t \\ &= \lambda \lim_{\eta\to 0}\frac{1}{\pi}\int_{E-\delta}^{E+\delta} \mathrm{Im}\frac{1}{t+\mathrm{i}\eta - \varepsilon(\mathbf{k})}\left\langle \chi_\Omega, \frac{1}{\widehat{H}_{\mathbf{k}} - t - \mathrm{i}\eta}\omega(0)\chi_\Omega\right\rangle \mathrm{d}t \\ &= \lambda\left\langle \chi_\Omega, \frac{1}{\widehat{H}_{\mathbf{k}} - \varepsilon(\mathbf{k})} f_\delta(\widehat{H}_{\mathbf{k}})\omega(0)\chi_\Omega\right\rangle, \end{aligned} \tag{20}$$

since $(t-\varepsilon(\mathbf{k}))^{-1}$ is continuous for $t \in [E-\delta, E+\delta]$. Using again that $f_\delta(\widehat{H}^o_{\mathbf{k}})\chi_\Omega = 0$, we find that the final term of Equation (20) equals

$$
\begin{aligned}
&= \left\langle \left[\frac{1}{\widehat{H}_{\mathbf{k}} - \varepsilon(\mathbf{k})} f_\delta(\widehat{H}_{\mathbf{k}}) - \frac{1}{\widehat{H}^o_{\mathbf{k}} - \varepsilon(\mathbf{k})} f_\delta(\widehat{H}^o_{\mathbf{k}}) \right] \chi_\Omega, \omega(0)\chi_\Omega \right\rangle \\
&= \lambda \lim_{\eta \to 0} \frac{1}{\pi} \int_{E-\delta}^{E+\delta} \frac{1}{t - \varepsilon(\mathbf{k})} \operatorname{Im} \frac{1}{t + \mathrm{i}\eta - \varepsilon(\mathbf{k})} \times \\
&\quad \times \left\langle \frac{1}{\widehat{H}_{\mathbf{k}} - t - \mathrm{i}\eta} \omega(0)\chi_\Omega, \omega(0)\chi_\Omega \right\rangle \mathrm{d}t \\
&= \lambda \left\langle \omega(0)\chi_\Omega, \frac{f_\delta(\widehat{H}_{\mathbf{k}})}{(\widehat{H}_{\mathbf{k}} - \varepsilon(\mathbf{k}))^2} \omega(0)\chi_\Omega \right\rangle.
\end{aligned} \tag{21}
$$

Putting Equations (20) and (21) together yields

$$
\begin{aligned}
\langle \chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}})\chi_\Omega \rangle &= \lambda^2 \langle \omega(0)\chi_\Omega, \frac{f_\delta(\widehat{H}_{\mathbf{k}})}{(\widehat{H}_{\mathbf{k}} - \varepsilon(\mathbf{k}))^2} \omega(0)\chi_\Omega \rangle \\
&\leqslant \lambda^2 \frac{1}{(|\varepsilon(\mathbf{k}) - E| - \delta)^2} \langle \omega(0)\chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}})\omega(0)\chi_\Omega \rangle.
\end{aligned}
$$

Thus, for any $\epsilon > \delta$,

$$
\begin{aligned}
&\int_{\{|\varepsilon(\mathbf{k}) - E| > \epsilon\}} \langle \chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}})\chi_\Omega \rangle \\
&\quad \leqslant \lambda^2 \frac{1}{(\epsilon - \delta)^2} \int_{T^d} \langle \omega(0)\chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}})\omega(0)\chi_\Omega \rangle \\
&\quad = \lambda^2 \frac{1}{(\epsilon - \delta)^2} \int_\Omega \omega(0)^2 \langle \delta_0, f_\delta(H_\omega)\delta_0 \rangle \mathrm{d}\mathbb{P}(\omega),
\end{aligned}
$$

where in the last equality we have inverted the Fourier transform, using Equation (17). We may estimate the right hand side with Hölder's inequality and the Wegner estimate:

$$
\begin{aligned}
&\int_\Omega \omega(0)^2 \langle \delta_0, f_\delta(H_\omega)\delta_0 \rangle \mathrm{d}\mathbb{P}(\omega) \\
&\quad \leqslant \|\omega(0)\|_q^2 \left(\int_\Omega \langle \delta_0, f_\delta(H_\omega)\delta_0 \rangle \mathrm{d}\mathbb{P}(\omega) \right)^{1 - \frac{2}{q}} \\
&\quad \leqslant \|\omega(0)\|_q^2 \left(\frac{\|\rho\|_\infty}{\lambda} 2\delta \right)^{1 - \frac{2}{q}},
\end{aligned}
$$

since $\langle \delta_0, f_\delta(H_\omega)\delta_0 \rangle^p \leqslant \langle \delta_0, f_\delta(H_\omega)\delta_0 \rangle$ for $p > 1$ (because $\langle \delta_0, f_\delta(H_\omega)\delta_0 \rangle \leqslant 1$). Here $\|\omega(0)\|_q^q = \int \omega(0)^q \mathrm{d}\mathbb{P}(\omega)$ for $q < \infty$ and $\|\omega(0)\|_\infty = \text{ess-sup}_\omega |\omega(0)|$.

Therefore

$$
\int_{T^d} \langle \chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}})\chi_\Omega \rangle \leqslant \Gamma(E)\epsilon + \lambda^2 \frac{1}{(\epsilon - \delta)^2} \|\omega(0)\|_q^2 \left(\frac{\|\rho\|_\infty}{\lambda} 2\delta \right)^{1 - \frac{2}{q}}, \tag{22}
$$

where the first term on the right hand side is an upper bound for

$$
\int_{\{|\varepsilon(\mathbf{k}) - E| \leqslant \epsilon\}} \langle \chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}})\chi_\Omega \rangle \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} \leqslant \int_{\{|\varepsilon(\mathbf{k}) - E| \leqslant \epsilon\}} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d}.
$$

Upon optimizing over $\epsilon \in (\delta, \infty)$, this implies

$$\int_{\Omega} \langle \delta_0, f_\delta(H_\omega)\delta_o \rangle \leqslant \Gamma(E)\delta + C_{\rho,q,\Gamma}\, \lambda^{\frac{1}{3}\left(1+\frac{2}{q}\right)} \delta^{\frac{1}{3}\left(1-\frac{2}{q}\right)},$$

which completes the proof of Theorem 1. □

3.2. PROOF OF THEOREM 3

If instead of being a regular point, E is a point of order α then the proof goes through up to Equation (22), in place of which we have

$$\int_{T^d} \langle \chi_\Omega, f_\delta(\widehat{H}_{\mathbf{k}})\chi_\Omega \rangle \leqslant \Gamma(E;\alpha)\epsilon^\alpha + \lambda^2 \frac{1}{(\epsilon-\delta)^2} \|\omega(0)\|_q^2 \left(\frac{\|\rho\|_\infty}{\lambda} \delta \right)^{1-\frac{2}{q}}.$$

Setting $\varepsilon = \delta + \lambda^\gamma \delta^\beta$ and choosing γ, β such that the two terms are of the same order yields

$$\gamma = \frac{1}{2+\alpha}\left(1+\frac{2}{q}\right), \quad \beta = \frac{1}{2+\alpha}\left(1-\frac{2}{q}\right),$$

which implies

$$\int_{\Omega} \langle \delta_0, f_\delta(H_\omega)\delta_o \rangle \leqslant \Gamma(E;\alpha)\delta^\alpha + C_q\, \lambda^{\frac{\alpha}{2+\alpha}\left(1+\frac{2}{q}\right)}\ \delta^{\frac{\alpha}{2+\alpha}\left(1-\frac{2}{q}\right)},$$

completing the proof. □

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