Minimal superadditive functions over an obstacle

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Lecture
 Definitions:

Let \( a = a_j = (a_j^1, ..., a_j^n) : \mathbb{N} \rightarrow \mathbb{R}^n \).

where \( a_j \) is a finite vector sequence of \( j \) (after some \( j \) all components are zero).
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where \( a_j \) is a finite vector sequence of \( j \) (after some \( j \) all components are zero). **Total value** is called the following vector

\[ \bar{a} = \sum a_j = (\sum a^1_j, ..., \sum a^n_j) \]
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Theorem 1

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Then

1. \( B(x) \) is defined on the set \( \sum_{m(\Omega)} \).
2. \( B(m(y)) \geq H(y) \) for all \( y \in \Omega \).
3. \( B(x) \) is superadditive.
4. \( B(x) \) is minimal among those who satisfy conditions 1, 2, 3.

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Proof, domain, item 1

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\( \text{Dom } B \subset \sum_{\infty} m(\Omega). \)

\( m(a_j) \in m(\Omega) \subset \sum_{\infty} m(\Omega), \forall j. \Rightarrow \sum m(a_j) \in \sum_{\infty} m(\Omega). \)
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\( \sum_{\omega} m(\omega) \subset \text{Dom } B. \ x \in \sum_{\omega} m(\omega), \implies x = \sum_{j=1}^{N} m(a_j), a_j \in \Omega \)
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Set $a_1^* = y$ and $a_j^* = 0$ for $j \geq 2$. 
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B(m(y)) = \sup_{a \in \Omega(\mathbb{N}): \sum m(a_j) = m(y)} \sum H(a_j) \geq \sum H(a_j^*) = H(y)
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3. $B(x + y) \geq B(x) + B(y)$
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Proof.

$\exists a, b \in \Omega(\mathbb{N})$ such that $\sum m(a_j) = x$, $\sum m(b_j) = y$ and

$$\sum H(a_j) > B(x) - \varepsilon, \quad \sum H(b_j) > B(y) - \varepsilon.$$
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Let $G$ satisfies these properties 1,2,3. Let’s recall

1. $G(x)$ is defined on the set $\sum_{m}^{\infty} m(\Omega)$.  

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Proof.

Let $G$ satisfies these properties 1,2,3. Let’s recall

1. $G(x)$ is defined on the set $\sum_{m=1}^{\infty} m(\Omega)$.
2. $G(m(y)) \geq H(y)$ for all $y \in \Omega$. 

We are done.
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Then $\forall a \in \Omega(\mathbb{N})$

$$\sum H(a_j) \leq \sum G(m(a_j)) \leq G \left( \sum m(a_j) \right) = G(x)$$
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If \( \sum_{\infty} m(\Omega) = m(\Omega) \) and \( R(x) \) is superadditive, then

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B(x) = R(x), \forall x \in m(\Omega)
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If \( B \) is homogeneous of degree 1, and it is defined on the convex cone then it is minimal concave function over an obstacle.
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Which function solves which problem?

Which problem can be solved by which function?

Thank you for your attention!