Integration with several variables.

Review: Integral of a single variable function

Definition
The definite integral of a function $f : [a, b] \rightarrow \mathbb{R}$, in the interval $[a, b]$ is the number

$$
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^n f(x_i^*) \Delta x,
$$

where $x_i^* \in [x_{i-1}, x_i]$ is called a sample point, while {$x_i$} is a partition in $[a, b]$, $i = 1, \ldots, n$, and with $x_0 = a$, $x_n = b$, and

$$
\Delta x = \frac{b-a}{n}.
$$

The integral as an area.

The sum $S_n = \sum_{i=1}^n f(x_i^*) \Delta x$ is called a Riemann sum. Then,

$$
\int_a^b f(x) \, dx = \lim_{n \to \infty} S_n.
$$

The integral $\int_a^b f(x) \, dx$ is the area in between the graph of $f$ and the horizontal axis.

Review: volume by slicing or rotation.

--- single variable

General problem: find the volume below a surface over a 2-D region.

Simple case: rectangular region.

Partition "P": Divide $[a, b]$ into $n$ subintervals:

$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$

$i$th interval: $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$

Riemann Sum:

$$
S_p = \sum_{i=1}^n A(p_i)(x_i - x_{i-1}) = \sum_{i=1}^n A(p_i)\Delta x_i
$$

: Sum Riemann 
, 2, 1], , [ : interval 
: lssubinterva into Divide : "Partition 
\text{Interval} 
\Delta x 
\text{Volume} 
\text{Area} 
\text{Rectangle} 
\text{Integral} 
\text{Volume} 
\text{Integration} 
\text{Rectangular} 
\text{Region} 
\text{Sum} 
\text{Riemann} 
\text{Limits} 
\text{Definite} 
\text{Integral} 
\text{Evaluation} 
\text{Integral}
Double integrals on rectangles

Definition

The double integral of a function \( f : R \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) in the rectangle \( R = [a, b] \times [c, d] \) is the number

\[
\iint_R f(x, y) \, dx \, dy = \lim_{n \to \infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i^*, y_j^*) \Delta x \Delta y
\]

where \( x_i^* \in [x_i, x_{i+1}] \)

\( y_j^* \in [y_j, y_{j+1}] \), are sample points,

while \( \{x_i\} \) and \( \{y_j\} \),

\( i, j = 0, \ldots, n \) are partitions of the intervals \([a, b]\) and \([c, d]\), and

\[
\Delta x = \frac{(b-a)}{n}, \quad \Delta y = \frac{(d-c)}{n}
\]

The double integral as a volume

The sum

\[
S_n = \lim_{n \to \infty} \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i^*, y_j^*) \Delta x \Delta y
\]

called a Riemann sum. Then,

\[
\iint_R f(x, y) \, dx \, dy = \lim_{n \to \infty} S_n
\]

The integral \( \iiint_R f(x, y) \, dx \, dy \) is the volume above \( R \) and below the graph of \( f \).

Idea of Fubini’s theorem

Computing each slice by integration with single variable
Fubini Theorem on rectangular domains

Theorem
If \( f : R \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \) is continuous in \( R = [a_0, a_1] \times [b_0, b_1] \), then
\[
\int \int_R f(x, y) \, dx \, dy = \int_{a_0}^{a_1} \left[ \int_{b_0}^{b_1} f(x, y) \, dy \right] \, dx.
\]
\[
= \int_{b_0}^{b_1} \left[ \int_{a_0}^{a_1} f(x, y) \, dx \right] \, dy.
\]

Remark: Fubini's Theorem: The order of integration can be switched in double integrals of continuous functions on a rectangle.

Notation: The double integral is also written as
\[
\int \int_R f(x, y) \, dx \, dy = \int_{a_0}^{a_1} \int_{b_0}^{b_1} f(x, y) \, dx \, dy.
\]

Example
Use Fubini's Theorem to compute the double integral
\[
\int \int_R f(x, y) \, dx \, dy = \int_0^3 \int_0^2 (xy^2 + 2x^2y^3) \, dx \, dy
\]
where \( f(x, y) = xy^2 + 2x^2y^3 \), and \( R = [0, 2] \times [1, 3] \). Integrate first in \( x \), then in \( y \).

Solution: Since \( x \in [0, 2] \) and \( y \in [1, 3] \),
\[
I = \int_0^3 \int_0^2 (xy^2 + 2x^2y^3) \, dx \, dy.
\]

We compute the interior integral in \( x \) first, keeping \( y \) constant. After that we compute the integral in \( y \).

Solution: We compute the integral in \( x \) first, keeping \( y \) constant.
\[
I = \int_0^3 \int_0^2 (xy^2 + 2x^2y^3) \, dx \, dy.
\]

Thus,
\[
I = \int_0^3 \left[ \int_0^2 (xy^2 + 2x^2y^3) \, dx \right] \, dy,
\]
\[
= \int_0^3 \left[ \frac{y^2}{2} x^2 \bigg|_0^2 + \frac{2y^3}{3} x^3 \bigg|_0^2 \right] \, dy,
\]
\[
= \frac{y^2}{2} \left[ \frac{8}{3} \right] + \frac{2y^3}{3} \left[ \frac{8}{3} \right],
\]
\[
= \frac{26}{3} + \frac{4}{3} \cdot 80 = \frac{372}{3}.
\]
Solution: Integrate first in $y$, then in $x$.

\[
I = \int_R f(x, y) \, dy = \int_1^3 \int_0^1 (xy^2 + 2x^2y^3) \, dy \, dx
\]

\[
I = \int_0^1 \left[ \int_1^3 (xy^2 + 2x^2y^3) \, dy \right] \, dx.
\]

\[
I = \int_0^1 \left[ \frac{1}{3} (x^3) + \frac{2}{4} (x^4) \right] \, dx.
\]

\[
= \int_0^3 \left[ \frac{26}{3} x + 40x^2 \right] \, dx = \frac{26}{3} x^2 + 40 \frac{1}{3} x^3.
\]

\[
I = \frac{26}{3} (2) + 40 \frac{8}{3} = \frac{372}{3}.
\]

A particular case of Fubini's Theorem

Corollary

If the continuous function $f: R \subset R^2 \to R$ satisfies that

\[
f(x, y) = g(x) h(y),
\]

then the double integral of function $f$ in the rectangle $R = [a_0, x_1] \times [y_0, y_2]$ is given by

\[
\int_{a_0}^{x_1} \int_{y_0}^{y_2} g(x) h(y) \, dy \, dx = \left( \int_{a_0}^{x_1} g(x) \, dx \right) \left( \int_{y_0}^{y_2} h(y) \, dy \right).
\]

Remark: In the case that $f(x, y)$ is a product of two functions $g$, $h$, with $g(x)$ and $h(y)$, then the double integral of $f$ is also a product of the integral of $g$ times the integral of $h$.

Example

Compute the double integral of $f(x, y) = \frac{1 + x^2}{1 + y^2}$ in the rectangular region $R = [0, 2] \times [0, 1]$.

Solution: $I = \int_R f(x, y) \, dx \, dy = \int_0^2 \int_0^1 \frac{1 + x^2}{1 + y^2} \, dy \, dx$.

\[
I = \left[ \int_0^2 \frac{1}{1 + y^2} \right] \left[ \int_0^1 (1 + x^2) \, dy \right].
\]

\[
I = \left( x^2 + \frac{1}{3} x^3 \right) \left( \text{arctan}(y) \right) \bigg|_0^1 = \left( 2 + \frac{8}{3} \right) \frac{\pi}{4} = \frac{14}{3} \frac{\pi}{4}.
\]

We conclude $\int_R f(x, y) \, dx \, dy = \frac{7}{6} \pi$.

Review: Area between curves

\[
\text{Area} = \int_a^b (f(x) - g(x)) \, dx
\]

where $[a, b]$ is given.

\[
\text{Area} = \int_{x_1}^{x_2} |g(x) - f(x)| \, dx
\]

where $x_i$ and $x_j$ are not given.

Need to solve $f(x) = g(x)$ for $x_i$ and $x_j$. 

Review: Fubini's Theorem on rectangular domains

**Theorem**

If \( f : R \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) is continuous in \( R = [a, b] \times [c, d] \), then

\[
\int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy.
\]

**Remark:** Fubini result says that double integrals can be computed doing two one-variable integrals.

**Remark:** On a rectangle is simple to switch the order of integration in double integrals of continuous functions.

Fubini's Theorem on Type I domains, \( y(x) \)

**Theorem**

If \( f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) is continuous in \( D \), then hold (Type I):

If \( D = \{(x, y) \in \mathbb{R}^2 : x \in [a_1(x), b_1(x)], y \in [g_1(x), g_2(x)]\} \), with \( g_1, g_2 \) continuous functions on \([a, b]\), then

\[
\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx = \int_a^b f(x, y) \, dy \, dx.
\]

Fubini's Theorem on Type II domains, \( x(y) \)

**Theorem**

If \( f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) is continuous in \( D \), then hold (Type II):

If \( D = \{(x, y) \in \mathbb{R}^2 : x \in [b_2(y), b_1(y)], y \in [g_1(y), g_2(y)]\} \), with \( h_1, h_2 \) continuous functions on \([c, d]\), then

\[
\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy = \int_c^d f(x, y) \, dx \, dy.
\]

Summary: Fubini's Theorem on non-rectangular domains

**Theorem**

If \( f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) is continuous in \( D \), then hold:

(a) (Type I) If \( D = \{(x, y) \in \mathbb{R}^2 : x \in [a_1, b_1], y \in [g_1(x), g_2(x)]\} \), with \( g_1, g_2 \) continuous functions on \([a, b]\), then

\[
\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx = \int_a^b f(x, y) \, dy \, dx.
\]

(b) (Type II) If \( D = \{(x, y) \in \mathbb{R}^2 : x \in [b_2(y), b_1(y)], y \in [c, d]\} \), with \( h_1, h_2 \) continuous functions on \([c, d]\), then

\[
\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy = \int_c^d f(x, y) \, dx \, dy.
\]
Example
Find the integral of \( f(x, y) = x^2 + y^2 \), on the domain \( D = \{ (x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, \ x^2 \leq y \leq x \}. \)

Solution:
This is a Type I domain, with lower boundary 
\[ y = g_1(x) = x^2, \]
and upper boundary 
\[ y = g_2(x) = x. \]

Solution: \[
\int \int_D f(x, y) \, dx \, dy = \int_0^1 \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx
\]
with \( g_1(x) = x^2 \) and \( g_2(x) = x \), we obtain
\[
I = \int_0^1 \int_{x^2}^{x} (x^2 + y^2) \, dy \, dx,
\]
\[
I = \int_0^1 \left[ x^2 \left( \frac{y^3}{3} \right) \right]_{x^2}^{x} \, dx
\]
\[
= \int_0^1 x^2 (x - x^2) + \frac{1}{3} (x^3 - x^6) \, dx.
\]

Rmk: For some problems, they can be solved either as Type I or Type II.

Example
Find the integral of \( f(x, y) = x^2 + y^2 \) on the domain \( D = \{ (x, y) \in \mathbb{R}^2 : 0 \leq x \leq \sqrt{y}, \ 0 \leq y \leq 1 \}. \)

Solution:
This is a Type II domain, with left boundary 
\[ x = h_1(y) = y, \]
and right boundary 
\[ x = h_2(y) = \sqrt{y}. \]

Remark:
This domain is both Type I and Type II: \( y = x^2 \Leftrightarrow x = \sqrt{y}. \)
Solution: \( I = \int_D f(x, y) \, dx \, dy = \int_0^1 \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy \)

with \( h_1(y) = y \) and \( h_2(y) = \sqrt{y} \), we obtain

\[
I = \int_0^1 \int_y^{\sqrt{y}} (x^2 + y^2) \, dx \, dy,
\]

\[
I = \int_0^1 \left[ \left( \frac{x^3}{3} \right)_y^{\sqrt{y}} + y^2 \left( x \right)_y^{\sqrt{y}} \right] \, dy,
\]

\[
I = \int_0^1 \left[ \frac{1}{3} (y^{3/2} - y^3) + y^2 (y^{1/2} - y) \right] \, dy.
\]

(Cont.)

Solution: \( I = \int_D f(x, y) \, dx \, dy = \int_0^1 \int_{g_1(y)}^{g_2(y)} f(x, y) \, dx \, dy \)

\[
I = \int_0^1 \int_{x_1}^{x_2} f(x, y) \, dx \, dy
\]

We conclude \( \int_D f(x, y) \, dx \, dy = \frac{3}{35} \).

(Cont.)

Domains Type I and Type II

Summary: We have shown that a double integral of a function \( f \) on the domain \( D \) given in the pictures below holds.

\[
\int_D f(x, y) \, dx \, dy = \int_0^1 \int_{g_1(y)}^{g_2(y)} f(x, y) \, dx \, dy = \int_0^1 \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy.
\]

Example

Find the limits of integration of \( \int_D f(x, y) \, dx \, dy \) in the domain

\( D = \{ (x, y) \in \mathbb{R}^2 | \frac{x^2}{4} + \frac{y^2}{9} \leq 1 \} \) when \( D \) is considered first as Type I and then as Type II.

Solution: The boundary is the ellipse \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \).

So, the boundary as Type I is given by

\[
y = 3\sqrt{1 - \frac{x^2}{4}} = g_1(x), \quad y = 2\sqrt{1 - \frac{x^2}{9}} = g_2(x).
\]

The boundary as Type II is given by

\[
x = 3\sqrt{1 - \frac{y^2}{9}} = h_1(y), \quad x = 2\sqrt{1 - \frac{y^2}{4}} = h_2(y).
\]
Calculating the area as the volume

Areas of a region on a plane

Definition
The area of a closed, bounded region $R$ on a plane is given by

$$A = \int_R dx \, dy.$$ 

Remark:
- To compute the area of a region $R$ we integrate the function $f(x, y) = 1$ on that region $R$.
- The area of a region $R$ is computed as the volume of a 3-dimensional region with base $R$ and height equal to 1.
Example
Find the area of \( R = \{(x, y) \in \mathbb{R}^2 : x \in [-1, 2], y \in [x^2, x+2]\} \).

Solution: We express the region \( R \) as an integral Type I, integrating first on vertical directions:

\[
A = \int_{-1}^{2} \int_{x^2}^{x+2} dy \, dx.
\]

Rmk.: This part is the set-up we learned in Calculus I.

\[
A = \int_{-1}^{2} \left[ y \right]_{x^2}^{x+2} dx = \int_{-1}^{2} (x+2-x^2) dx = \left[ \frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2}
\]

\[
A = 2 - \frac{1}{2} + 4 + 2 - \frac{8}{3} - \frac{1}{3} = 8 - \frac{1}{2} - 3 \quad \Rightarrow \quad A = \frac{9}{2}.
\]

\( \square \)

Average value of a function

Review: The average of a single variable function.

Definition
The average of a function \( f : [a, b] \to \mathbb{R} \) on the interval \([a, b]\), denoted by \( \bar{f} \), is given by

\[
\bar{f} = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx.
\]

Definition
The average of a function \( f : R \subset \mathbb{R}^2 \to \mathbb{R} \) on the region \( R \) with area \( A(R) \), denoted by \( \bar{T} \), is given by

\[
\bar{T} = \frac{1}{A(R)} \int_{R} f(x, y) \, dx \, dy.
\]

Example
Find the average of \( f(x, y) = xy \) on the region
\( R = \{(x, y) \in \mathbb{R}^2 : x \in [0, 2], y \in [0, 3]\} \).

Solution: The area of the rectangle \( R \) is \( A(R) = 6 \).

We only need to compute
\[
I = \int_{R} f(x, y) \, dx \, dy
\]

\[
I = \int_{0}^{2} \int_{0}^{3} xy \, dx \, dy = \int_{0}^{2} x \left( \frac{x^2}{2} \right)_{0}^{3} \, dx = \int_{0}^{2} \frac{9x}{2} \, dx
\]

\[
I = \frac{9}{2} \left( \frac{x^2}{2} \right)_{0}^{2} = I = 9.
\]

Since \( \bar{T} = I/A(R) = \frac{9}{6} \), we get \( \bar{T} = \frac{3}{2} \).
Example
Find the integral of $\rho(x, y) = x + y$ in the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$.

Solution: We need to compute
\[ M = \int_{R} \rho(x, y) \, dx \, dy. \]
Remark: If $\rho$ is the mass density, then $M$ is the total mass.
\[ M = \int_{0}^{1} \int_{0}^{2x} (x + y) \, dy \, dx = \int_{0}^{1} \left[ x \left( y^2 \right)_{0}^{2x} + \left( \frac{y^2}{2} \right)_{0}^{2x} \right] \, dx. \]
\[ M = \int_{0}^{1} \left[ 2x^3 + 2x^2 \right] \, dx = 4 \cdot \frac{x^4}{3} \bigg|_{0}^{1} \Rightarrow M = \frac{4}{3} \quad <1. \]

Example
Given the function $\rho(x, y) = x + y$, the number $M$ computed in the previous example, and the triangle with boundaries $y = 0$, $x = 1$ and $y = 2x$, find the numbers
\[ \bar{r}_x = \frac{1}{M} \int_{R} x \rho(x, y) \, dy \, dx, \quad \bar{r}_y = \frac{1}{M} \int_{R} y \rho(x, y) \, dx \, dy. \]
Remark: $r = (\bar{r}_x, \bar{r}_y)$ is the center of mass of the body.
Solution: Recall: $M = \frac{4}{3}$. We need to compute
\[ \bar{r}_x = \frac{1}{M} \int_{0}^{1} \int_{0}^{2x} (x + y) \, dy \, dx = \frac{3}{4} \left[ \int_{0}^{1} x^2 \left( y^2 \right)_{0}^{2x} + \left( \frac{y^2}{2} \right)_{0}^{2x} \right] \, dx \]
\[ \bar{r}_y = \frac{3}{4} \int_{0}^{1} \left[ 2x^3 + 2x^2 \right] \, dx = \frac{3}{4} \cdot \frac{x^4}{1} \bigg|_{0}^{1} \Rightarrow \bar{r}_y = \frac{3}{4}. \]

(Continue)
\[ \bar{r}_x = \frac{1}{M} \int_{0}^{1} \int_{0}^{2x} (x + y) \, dy \, dx = \frac{3}{4} \int_{0}^{1} \left[ x \left( y^2 \right)_{0}^{2x} + \left( \frac{y^2}{2} \right)_{0}^{2x} \right] \, dx \]
\[ \bar{r}_y = \frac{3}{4} \int_{0}^{1} \left[ 2x^3 + 2x^2 \right] \, dx = \frac{3}{4} \cdot \frac{x^4}{1} \bigg|_{0}^{1} \Rightarrow \bar{r}_y = \frac{3}{4}. \]
\[ \Rightarrow (\bar{r}_x, \bar{r}_y) = \left( \frac{1}{4}, \frac{3}{4} \right) \quad \text{Center of the mass.} \]

Bacterium population.
If $f(x, y) = \frac{10,000e^{y}}{1+|x|^{1/2}}$ represents the population density of a certain bacterium on the $xy$-plane where $x$ and $y$ are measured in centimeters, find the total population of bacteria within the rectangle: $-5 \leq x \leq 5, -2 \leq y \leq 0$. 

10/23/2013
Bacterium population.

If \( f(x, y) = \frac{10,000e^y}{1 + x/2} \) represents the population density of a certain bacterium on the xy-plane where \( x \) and \( y \) are measured in centimeters, find the total population of bacteria within the rectangle: \(-5 \leq x \leq 5, -2 \leq y \leq 0\).

Solution: Total population = \( \iint \text{density} \, dA \)

\[
= \int_{-5}^{5} \int_{-2}^{0} \frac{10,000e^y}{1 + |x|/2} \, dy \, dx = \int_{-5}^{5} \frac{10,000}{1 + |x|/2} (1 - e^{-2}) \, dx \\
= 10^4 (1 - e^{-2}) \left[ \int_{-5}^{0} \frac{1}{1 - x/2} \, dx + \int_{0}^{5} \frac{1}{1 + x/2} \, dx \right] \\
= 10^4 (1 - e^{-2}) \left[ -2\ln \left| 1 - \frac{x}{2} \right| \bigg|_{-5}^{0} + 2\ln \left| 1 + \frac{x}{2} \right| \bigg|_{0}^{5} \right] \\
= 4 \times 10^4 (1 - e^{-2}) \times \ln \frac{7}{2} \approx 43329
\]