The Chain Rule

If \( w = f(x) \) and \( x = g(t) \) then,

\[
\frac{dw}{dt} = \frac{dw}{dx} \frac{dx}{dt}
\]

but what happens if we have a function of several variables:

\( w = f(x, y) \) and \( x = g(t), \ y = h(t) \)

then,

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}
\]
Example:

\[ w = x^2 + y^2, \quad x = \cos t, \quad y = t^2 \]

Find \( \frac{dw}{dt} \)

Solution:

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} =
\]

\[
= (2x)(-\sin t) + (2y)(2t) =
\]

\[
= (2 \cdot \cos t)(-\sin t) + (2t^2)(2t) =
\]

\[
= -2 \cos t \cdot \sin t + 4t^3.
\]

What if \( w(x,y,t) = ? \)

\[
\begin{align*}
\text{What if} \quad w(x,y,t) & = ? \\
& x = (0,5)
\end{align*}
\]
Assume

\[ w = f(x, y, z) \quad \text{and} \]

\[ x = g(s, t), \quad y = h(s, t), \quad z = r(s, t) \]

Then

\[
\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \\
+ \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}
\]

and

\[
\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t} + \\
+ \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial t}
\]
Example,

\[ W = x^2 + y^2 + z^2, \quad x = s + t, \quad y = s - t \]

\[ z = s \cdot t \]

Find

\[ \frac{dw}{ds} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s} = \]

\[ = (2x) \cdot 1 + (2y) \cdot 1 + (2z) \cdot t = \]

\[ = 2x + 2y + 2z \cdot t = 2(s + t) + 2(s - t) + 2s \cdot t \]

\[ + 2s \cdot t \cdot t = 2s + 2t + 2s - 2t + 2st^2 = \]

\[ = 4s + 2st^2 \]