LECTURE 4

Section 14.1

Functions of several variables

\[ f(x_1, x_2, \ldots, x_n) \to \mathbb{R} \]

Example: \[ f(x_1, x_2) = x_1^2 + x_2^2 \]

\[ f(x, y) = (\sin x + \cos x) \cdot xy \]

Domain and range

**Domain** - for which variables the function is well defined.

Example: \[ f(x, y) = \sqrt{x^2 - y} \quad \text{for} \quad x^2 \geq y \]

Domain = \{ (x, y) : x^2 \geq y \}

Example \[ f(x, y) = \ln(x + y) \]

Domain = \{ (x, y) : x + y \geq 0 \}
Range of a function

Range: What values does a function take?

Example:

1. \( f(x, y) = x^2 + y^2 \). Range is \([0, \infty)\)

2. \( f(x) = \ln(x + y) \). Range is \((-\infty, \infty)\)

3. \( f(x, y, z) = 1 + \cos x + \cos y + \cos z \)

   Range is \([-2, 4]\) (Why? Because \(-1 \leq \cos x \leq 1\))

Interior points, boundary points, and bounded domains.

Assume you have a set in 2D (or 3D) \( \mathbb{R}^2 \) (or \( \mathbb{R}^3 \)) \( \subseteq \mathbb{R}^2 \) it does not matter
What are boundary points? This is exactly boundary of this domain i.e. (but not the points which are inside)

2. What are interior points?
This is a set without boundary does not have boundary.
What is bounded domain (set)?

Answer: Which is bounded i.e. can be included in a ball of finite radius.

Example

[Diagram]

← this is not bounded

These are bounded
More examples:

1. Lines are unbounded
2. Planes are unbounded

Remark: Interior points can be characterized in the following way as well:

If you can find a ball which contains this point and is contained inside this domain, then the point is interior point.

Example: Consider $\mathbb{R}^n$ without the origin.

What are interior points? Answer: everything.
Graphs, level curves of a function

Assume you have function \( f(x) \) (of one variable). What is a graph of this function?

Answer:

This is a graph of \( f(x) \) and these kind of points \((x, f(x))\).

If \( f(x,y) \), then this is a graph of \((x,y,f(x,y))\) points.
What is a level curve?

It is an intersection of the graph (surface) and the plane \( z = c \) when \( c \) is constant.

Example:

What is a level curve when \( z = 1 \)?

You do the following:

This intersection is a level curve.

Or

\[ z = 1 \]
For different values of $z$, you get different level curves.
If you have \( f(x,y) \)
And the question is sketch the level curve you should do the following.

1) You should construct curves \( f(x,y) = c \) for different values of \( c \).

Example:

\[
f(x, y) = x^2 + y^2
\]

Q: Sketch the level curves

Solution: \( f(x,y) = 1 \), \( x^2 + y^2 = 1 \)

Circle of radius 1.
If you have a function of 3 variables $f(x,y,z)$, and the question is sketch the level surface, then you do the following...
You should draw a graph $f(x,y,z) = c$ for different values of $c$.

**Example:**

$f(x,y,z) = x^2 + y^2 + z^2$.

2. Sketch the level surfaces

**Solution:**

$f(x,y,z) = 1$ or $x^2 + y^2 + z^2 = 1$

Sphere of radius 1

$f(x,y,z) = 2$ or $x^2 + y^2 + z^2 = 2$

Sphere of radius $\sqrt{2}$

and so on.
limits and continuity of a function in several variables

What does the following expression mean?

\[ \lim_{(x,y) \to (x_0,y_0)} f(x,y) \]

It says: What is the value of \( f(x,y) \) as \( (x,y) \) approaches \( (x_0,y_0) \)?

Example:

\[ \lim_{(x,y) \to (0,0)} x^2 + y^2 = 0. \]
The following \((x,y) \to (0,0)\) means that \(x \to 0\) and \(y \to 0\).

**Example:**

\[
\lim_{(x,y) \to (1,2)} \frac{x+y}{\ln y} = \frac{1+2}{\ln 2} = \frac{3}{\ln 2}
\]

Sometimes this value

\[
\lim_{(x,y) \to (x_0,y_0)} f(x,y)
\]

might not exist at all!

**Example:**

\[
\lim_{(x,y) \to (0,0)} \frac{1}{x+y}
\]
does not exist.
because if we approach to (0,0) from "negative side" like 
\( x \to 0 \) but \( x < 0 \) and \( y \to 0 \) 
but \( y > 0 \), for example,

\[
x = -\frac{1}{1}, -\frac{1}{0.1}, -\frac{1}{0.01}, -\frac{1}{0.001}, \ldots \to 0
\]

and

\[
y = -\frac{1}{1}, -\frac{1}{0.1}, -\frac{1}{0.01}, -\frac{1}{0.001}, \ldots \to 0
\]

They approach to zero but they are always negative then 
\( x + y < 0 \) as well an therefore

\[
\frac{1}{x + y} \approx \frac{1}{-\infty} = -\infty
\]

\( x + y \) something small negative
On the other hand

If we approach from the "positive side" like

\[ x = 0.1, \quad 0.001, \quad 0.0001, \ldots \]

\[ y = 0.1, \quad 0.001, \quad 0.0001 \]

Then \( x, y \to 0 \) but \( x \cdot y > 0 \)

Therefore \( x + y > 0 \) and Hence

\[ \frac{1}{x + y} = \frac{1}{\text{something small positive}} = \infty \]

So it takes two different values \(-\infty\) and \(+\infty\)
If these values always coincide regardless how do we approach to the point then we say that the limit exists.

Otherwise we say that limit does not exist.

Example when limit does not exist

\[
\lim_{(x,y) \to (0,0)} \frac{x+y^2}{x+y}
\]

Solution: Let \( y = 0 \) and \( x \to 0 \)

then \( \lim_{(x,y) \to (0,0)} \frac{x+y^2}{x+y} = \lim_{x \to 0} \frac{x}{x} = \lim_{(x,0) \to (0,0)} 1 = 1 \)
Now let \( x = 0 \) and \( y \to 0 \) then

\[
\lim_{(x,y) \to (0,0)} \frac{x + y^2}{x + y} = \lim_{(x,y) \to (0,0)} \frac{y^2}{y} = \lim_{(x,y) \to (0,0)} y = 0
\]

but \( 1 \neq 0 \) so limit does not exist.

Example

\[
\lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2}
\]

Let's show that limit does not exist.

Indeed let's take \( x = ky \) and \( y \to 0 \). Then of course \( x \to 0 \) as well.
\[
\lim_{(x,y) \to (0,0)} \frac{xy}{x^2+y^2} = \lim_{(ky,y) \to 0} \frac{ky^2}{k^2y^2+y^2} = \\
= \lim_{(ky,y) \to (0,0)} \frac{k}{k^2+1} = \frac{k}{k^2+1}
\]

but for different values of \( k \) the answer is different. Hence the limit does not exist.

**Example:**

\[
\lim_{(x,y) \to (0,0)} \frac{x^2-y^2}{x-y} = \lim_{(x,y) \to (0,0)} \frac{(x-y)(x+y)}{x-y} = \\
= \lim_{(x,y) \to (0,0)} x+y = 0. \quad \text{**Exists!**}
\]
If the limit exists and moreover

\[ \lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0) \]

Then we say that the function is continuous.

Example:

The function \( f(x, y) = \frac{x^2 - y^2}{x - y} \) is undefined at point \((x, y) = (0, 0)\). If you define its value to be 0 like \( f(0, 0) = 0 \), then it is continuous; otherwise it is not.