lines in 3D.

In order to determine a line in 3D, we need to know the direction of this line and a point through which it passes.

\[ \vec{V} = \langle v_1, v_2, v_3 \rangle \]

\( \vec{V} \) - direction (any vector)

\[ (x_0, y_0, z_0) \]

\((x, y, z)\) such that

\[ \begin{aligned}
  x &= x_0 + t \cdot v_1 \\
  y &= y_0 + t \cdot v_2 \\
  z &= z_0 + t \cdot v_3
\end{aligned} \]

\(-\infty < t < \infty \), \( t \) - is a parameter

Vector equation: \( \langle x, y, z \rangle = \)

\[ \langle x_0 + tv_1, y_0 + tv_2, z_0 + tv_3 \rangle = \langle x_0, y_0, z_0 \rangle + \langle tv_1, tv_2, tv_3 \rangle = \]

\[ = \langle x_0, y_0, z_0 \rangle + t \cdot \langle v_1, v_2, v_3 \rangle = \vec{v}_0 + t \cdot \vec{V} \]

where \( \vec{v}_0 = \langle x_0, y_0, z_0 \rangle \)
If instead of \( \mathbf{J} \) we consider \( 10 \mathbf{J} \) then we get the same equation of the line.

\[
\mathbf{r}(t) = \mathbf{r}_0 + t \cdot \mathbf{v}
\]

Our magic formula

Example: let \( \mathbf{r}_0 = \langle 1, 0, 1 \rangle \) and \( \mathbf{v} = \langle 0, 1, 0 \rangle \)

then \( \mathbf{r}(t) = \mathbf{r}_0 + t \cdot \mathbf{v} = \langle 1, 0, 1 \rangle + t \cdot \langle 0, 1, 0 \rangle = \langle 1, t, 1 + t \rangle \) \(-\infty < t < \infty\)
Example:
Find the equation of a line which passes through the points $A = (x_0, y_0, z_0)$, $B = (x_1, y_1, z_1)$.

Solution - direction is obtained by the vector $\overrightarrow{AB} = (x_1-x_0, y_1-y_0, z_1-z_0)$.

For the initial point let's pick $A = (x_0, y_0, z_0)$.

Therefore $\vec{r}(t) = \vec{r}_0 + t \cdot \vec{V} = \overrightarrow{OA} + t \cdot \overrightarrow{AB} = (x_0, y_0, z_0) + t \cdot (x_1-x_0, y_1-y_0, z_1-z_0)$.

Important formula
Remark: If \( \mathbf{A} = (x_0, y_0, z_0) \) is a point and we want to create a vector with starting point 0 and endpoint \( \mathbf{A} \), then we just write:

\[
\overrightarrow{OA} = (x_0, y_0, z_0)
\]

If \( \mathbf{r}(t) = \mathbf{r}_0 + t \cdot \mathbf{v} \), then \( \mathbf{v} \) is called velocity of the parametrisation of the line.

If we consider

\[
\mathbf{v}(t) = \mathbf{v}_0 + t \cdot \frac{\mathbf{v}}{||\mathbf{v}||}
\]

then it is the same line (direction is the same), but magnitude of velocity is different.
The distance from a point to a line in 3D.

\[ A = (x_0, y_0, z_0) \]

\[ d = ? \]

\[ \mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v} \cdot t \]

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How to derive a formula?

**Solution:**

1. Pick any point on the line, say \( P \).

2. Create a vector \( \overrightarrow{PA} \).

Let \( \theta \) be an angle between \( \overrightarrow{PA} \) and \( \mathbf{v} \).

Then distance:

\[ d = \left\| \overrightarrow{PA} \right\| \cdot \sin \theta = \frac{\left| \overrightarrow{PA} \cdot \mathbf{v} \right|}{\left\| \mathbf{v} \right\|} \]

\[ = \frac{\left| \overrightarrow{AP} \times \mathbf{v} \right|}{\left\| \mathbf{v} \right\|} \]
Thus: \[ \text{distance} = \frac{\| \overrightarrow{PA} \times \mathbf{u} \|}{|\mathbf{u}|} \]

Example: Let \( \overline{r}(t) = <1,1,1> + t \cdot <1,2,3> \)
and let \( A = <0,0,0> \). Find the distance between \( A \) and the line \( \overline{r}(t) \).

Solution: Let's choose \( \mathbf{p} = <1,1,1> \).

\[ \text{distance} = \frac{\| \overrightarrow{PA} \times \mathbf{u} \|}{|\mathbf{u}|} \]

Then \( \overrightarrow{PA} = <0,0,0> - <1,1,1> = <-1,-1,-2> \)

\[ \overrightarrow{PA} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & -1 \\ 1 & 2 & 3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -1 & -1 \\ 1 & 3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} \]

\[ \overrightarrow{PA} \times \mathbf{u} = \mathbf{i} \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} = \mathbf{i}(-2 - (-1)) = \mathbf{i}(-1) = -\mathbf{i} \]

Thus, the distance is 1 unit.
\[ \vec{PA} \times \vec{v} = i(-3+2) - j(-3+2) + k(-2+1) = \]
\[ = -i + j - k = \langle -1, 1, -1 \rangle \]

Therefore, \[ |\vec{PA} \times \vec{v}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} \]
\[ |\vec{v}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14} \]

Hence, distance = \[ \frac{|\vec{PA} \times \vec{v}|}{|\vec{v}|} = \]
\[ = \frac{\sqrt{3}}{\sqrt{14}} \]

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Equation of a Plane
The set of points which satisfy the following equation

\[ Ax + By + Cz = D \]

is a plane

(Another way of writing this equation is \[ A(x-x_0) + B(y-y_0) + C(z-z_0) = 0. \]
This is a plane which passes through the point \((x_0, y_0, z_0)\). The numbers \(A, B, C\) create a vector \(\langle A, B, C \rangle\) which is perpendicular to a plane.

\[ \mathbf{n} = \langle A, B, C \rangle \]

\[ A(x-x_0) + B(y-y_0) + C(z-z_0) = 0 \]

Sometimes \(\langle A, B, C \rangle\) is called the normal vector.

**Example:** Let \(2x-z = 0\). Find a vector which is perpendicular to the plane.

**Solution:** \(\mathbf{n} = \langle 2, 0, -1 \rangle\)
Example (Important)

Let \( A = \langle 0,0,0 \rangle \), \( B = \langle 0,1,2 \rangle \), \( C = \langle 1,2,2 \rangle \).

Q. Find the equation of a plane which contains the points \( A, B, C \).

Solution:

1 step: we need to find a vector which is perpendicular to a plane.
Let's create a vector \( \overrightarrow{AB} \) and \( \overrightarrow{AC} \). Then we know that \( \overrightarrow{AB} \times \overrightarrow{AC} \) is a vector which is perpendicular to \( \overrightarrow{AB} \) and \( \overrightarrow{AC} \) and hence it is perpendicular to the plane.

\[
\overrightarrow{AB} = \langle 0, 1, 2 \rangle \\
\overrightarrow{AC} = \langle 1, 2, 2 \rangle
\]

\[
\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{vmatrix} = -2i + 2j - k = \langle -2, 2, -1 \rangle
\]

So instead of normal vector \( \overrightarrow{n} \), we can choose \( \overrightarrow{n'} = \langle -2, 2, -1 \rangle \).
2 step: we need to find a point which belongs to a plane. For example choose $A = (0,0,0)$ (Any other choices works as well).

Therefore

$$-2(x-0) + 2(y-0) + (z-0) = 0$$

or $-2x + 2y - z = 0$.

The distance between a point and the plane.

$$A = (x_0, y_0, z_0)$$

$d = ?$ 

$$Ax + By + Cz = D.$$
Pick any point in the plane. Say point $P$.

$$\vec{n} = \langle A, B, C \rangle$$

Create a vector $\vec{PA}$. Let's project $\vec{PA}$ onto vector $\vec{n}$.

So consider $\text{Proj}_\vec{n} \vec{PA} = \left( \frac{\vec{PA} \cdot \vec{n}}{|\vec{n}|^2} \right) \vec{n}$. It is clear that $|\text{Proj}_\vec{n} \vec{PA}| = \text{distance}$.

Therefore:

$$|\text{Proj}_\vec{n} \vec{PA}| = \left| \frac{\vec{n} \cdot \vec{PA}}{|\vec{n}|^2} \right| |\vec{n}| = \frac{|\vec{n} \cdot \vec{PA}|}{|\vec{n}|} = \frac{|\vec{n} \cdot \vec{PA}|}{|\vec{n}|}$$
Hence
\[ \text{distance} = \frac{|\vec{n} \cdot \vec{PA}|}{|\vec{n}|} \]

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Example:

\[ x + y + z = 0 \]

be a plane

and let \( A = (1, 1, 1) \)

Find the distance between \( A \) and the plane:

Solution:

Pick any point which belongs to a plane, say \((0, 0, 0) = P\)

then \( \vec{PA} = <1, 1, 1> \), \( \vec{n} = <1, 1, 1> \)

\[
\text{distance} = \frac{|\vec{PA} \cdot \vec{n}|}{|\vec{n}|} = \frac{|1+1+1|}{\sqrt{1^2+1^2+1^2}} = \frac{3}{\sqrt{3}} = \sqrt{3}
\]
Angles between planes

Let \( x + y + 2z = 0 \) are two planes

Find the angle between these planes.

Solution: \( \mathbf{n}_1 = \langle 1, 1, 1 \rangle \) is a perpendicular vector to the first plane. \( \mathbf{n}_2 = \langle 1, 1, 2 \rangle \) is a perpendicular vector to the second one.

The angle between these vectors equals to the angle between planes.

\[
\theta = \arccos \left( \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right)
\]

\[
\cos \theta = \frac{n_1 \cdot n_2}{|n_1| |n_2|}
\]
Therefore

\[ \theta = \arccos \left( \frac{n_1 - n_2}{|n_1| \cdot |n_2|} \right) = \]

\[ = \arccos \left( \frac{1 + 1 + \lambda}{\sqrt{1 + 1 + 1} \cdot \sqrt{1^2 + 1^2 + 2^2}} \right) = \]

\[ = \arccos \left( \frac{\lambda}{\sqrt{3} \cdot \sqrt{6}} \right) = \arccos \left( \frac{\lambda}{\sqrt{18}} \right) \]
Example:

Let \( x+y=0 \) be two planes
\[ 2x+z = 0 \]

Find the equation of the line of intersection

Solution: First way:

Pick any point from the intersection

\[
\begin{cases}
  x+y = 0 \\
  2x+z = 0
\end{cases}
\]

Hence \( x = -y \) 
Hence \( -2y+z = 0 \)
So \((-1,1,2)\) works
\( \vec{h}_1 = (1, 4, 0) \) is perpendicular to the first plane.
\( \vec{h}_2 = (2, 0, 1) \) is perpendicular to the second plane.

Then the direction of a line is given by a vector \( \vec{v} \) which is perpendicular both to \( \vec{h}_1 \) and \( \vec{h}_2 \).

Therefore we can choose
\[ \vec{v} = \vec{h}_1 \times \vec{h}_2 \]
\[ \mathbf{n} \times \mathbf{n} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = i - 2j - 2k = \langle 1, -2, -2 \rangle \]

Therefore \( \mathbf{r}(t) = \langle -1, 4, 2 \rangle + t \cdot \langle 1, -2, -2 \rangle \)

2nd way: Pick any points (2 points) which belong to the intersection. Say \( \mathbf{A}, \mathbf{B} \). Then construct a line which passes through those points.