Recall: If we have two points $A$ and $B$ in 3D and we have a path $c$ (or curve) joining these two points, and also we have a vector field $F = \langle M, N, P \rangle$, then we learned how to compute $\int_C F \cdot dr$ - work, circulation, flow.

$$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot \frac{dr}{dt} \, dt$$

and $r(t)$ is a parameterization of a curve $C$. 
But do we have a formula like:

\[ \int_{A}^{B} F \cdot dr = \text{surf}(B) - \text{surf}(A) \]

like in 1-dimensional case?

In general we don't have 😞

But we do have this kind of formula if: there exists a scalar function \( f(x,y,z): \mathbb{R}^3 \rightarrow \mathbb{R} \) such that \( \nabla f = F \). In this case

\[ \int_{A}^{B} F \cdot dr = f(B) - f(A) \]

Regardless of the choices of the path \( C \) joining the points \( A \) and \( B \).
This means that integral does not depend on path. So it is path independent.

Example: Let $A = (0, 0, 0)$, $B = (1, 1, 1)$ and $F = \langle yz, xz, xy \rangle = yz \cdot i + xz \cdot j + xy \cdot k$

Find $\int_{A} F \cdot dr$

Solution:

Note that if $f(x, y, z) = xyz$ then

$\nabla f = \langle f_x, f_y, f_z \rangle = \langle yz, xz, xy \rangle = F$

Hence

$$\int_{(0,0,0)}^{(1,1,1)} F \cdot dr = \int_{(0,0,0)}^{(1,1,1)} \nabla f \cdot dr = f(1,1,1) - f(0,0,0) = 1.$$
We solved the problem!

But how do we find $f$?

And how to check if that $f$ existed?

If such $f$ exists then the vector field $F$ is called \textit{conservative}.

Question: When does such $f$ exist?

Answer: Here is the test:

Let $F = \langle M, N, P \rangle$

If \[
\frac{\partial F}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial L}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}\]

Then such $f$ exist (so the vector field $F$ is \textit{conservative}).
How to remember this formula?

We want \( f \) such that

\[
\langle f_x, f_y, f_z \rangle = \langle M, N, P \rangle
\]

Or,

\[
\begin{align*}
  f_x &= M \\
  f_y &= N \\
  f_z &= P
\end{align*}
\]

but \( f_{xy} = f_{yx} \), \( f_{xz} = f_{zx} \) and \( f_{yz} = f_{zy} \), therefore

for example, this \( f_{xy} = f_{yx} \) implies that

\[
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
\]

and so on you can get others as well.

But how to find such \( f \) -?

Here is the solution on some particular example:
Example:

Let \( F = (2x+y+z)i + (2y+x+z)j + (2z+y+x)k \)

Find \( f \) such that \( \nabla f = F \)

Solution: if \( \nabla f = F \) it means

\[ \langle f_x, f_y, f_z \rangle = \langle 2x+y+z, 2y+x+z, 2z+y+x \rangle \]

So it means:

1. \( f_x = 2x+y+z \)
2. \( f_y = 2y+x+z \)
3. \( f_z = 2z+y+x \)

\[ f_x = 2x+y+z \] integrate over \( x \) we get

\[ f = \int (2x+y+z) dx = x^2+yx+zx+C \]

when constant \( C \) depends on \((y,z)\) So

\[ f = x^2+yx+zx+C(y,z) \]

So we still need to find \( C(y,z) \)

Let's use 2 and 3.
So \( f = x^2 + yx + z \)
but \( f_y = 2y + x + 2 \) therefore

\[ \left( x^2 + yx + z + C(y,z) \right)' \bigg|_y = 2y + x + 2 \]

\( x + C_y'(y,z) = 2y + x + 2 \) on

\( C_y'(y,z) = 2y + x + 2 = 2y + 2 \)

Now let's integrate over \( y \).

\[ C(y,z) = \int (2y + 2) \, dy = \int (2y + 2) \, dy \]

\( y^2 + 2y + B \), where \( B \) is some constant
which depends on \( z \). So \( B = B(x) \)

Let's collect collect (1) and (2)

(1) implies \( f = x^2 + yx + z + C_y(z) \), (2) implies \( C(y,z) = y^2 + 2y + B(x) \)

Hence

\[ f = x^2 + yx + z + y^2 + 2y + B(z) \]
but still we need to find \( B(x) \)? Let's use \( \text{circ} \) or \( \text{arc} \).

Recall: \( f = x^2 + yx + zx + y^2 + 2y + B(z) \)

\( \text{circle} \)

\( \text{arc} \)

3 \( \int z = 2z + y + x \)

\( x^2 + yx + zx + y^2 + 2y + B(z) \int \frac{d}{dz} = 2z + y + x \)

\( x + y + B'(z) = 2z + y + x \)

\( B'(z) = 2z \)

\( B(z) = \int 2z \, dz = \frac{z^2}{2} + C \)

which does not depend on anything

So

\( f(x, y, z) = \int x^2 + yx + zx + y^2 + ty + z^2 \)

Remark: Such function \( f \) is called potential function for vector field \( F \).
Greens formula

[flux, circulation]

Sometimes is called outward flux

Let $C$ be closed simple curve $\partial D$

Then we know that

$$\text{flux} = \int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_C F(r(t)) \cdot \left( \frac{dy}{dt} \frac{dx}{dt} \right) dt$$

This is the way how do we compute this flux.

But there is a simple way (Greens formula)

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \left( \frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} \right) \, dx \, dy$$

\text{interior of a curve (everything inside!)}
So one integral becomes double integral over the domain enclosed by the curve C.

Circulation:

\[ \text{Circulation} = \oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy \]

counter clockwise

domain enclosed by a curve C.

Examples:

Let \( \mathbf{F}(x,y) = (\cot(\omega x) + y, \cot(\omega y) - x) \)

and let find the circulation over the boundary of \( y = x^2 \) and \( y = 2x \) domain bounded by these curves.
Solution:

\[
\begin{align*}
\text{Circulation} &= \iint_S \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy \\
M &= \langle \tan (\cos x) + y, \cot (\sin y) - x \rangle \\
N &= 1 \\
\end{align*}
\]

\[
\int_{-1}^{1} \int_{-1}^{1} [-1 - (+1)] \, dx \, dy = -2 \cdot \iint_S dx \, dy = -2 \cdot \iint_S dy \, dx
\]

\[
\begin{align*}
-2 \int_{0}^{2} \int_{0}^{2x} 1 \, dy \, dx &= -2 \int_{0}^{2} (2x - x^2) \, dx = -2 \left[ x^2 - \frac{x^3}{3} \right]_0^2 \\
&= -2 \left[ 4 - \frac{8}{3} \right] = \left[ -\frac{8}{3} \right]
\end{align*}
\]