Triple integrals in Cylindrical and Spherical coordinate system

Section 15.7

Instead of \((x, y, z)\) you use \((r, \theta, z)\).

- \(z\) is a height, \(r\) is distance between Origin and Projection of a point \(P\) onto \(XY\) plane and \(\theta\) is an angle between \(X\) axis and the projected point (counterclockwise).

How do we integrate in cylindrical coordinate system?
The function might be given as $f(x, y, z)$ or $f(r, \theta, z)$

**Example:**

$f(x, y, z) = x^2 + y^2 + z^2$

**Example:**

$f(r, \theta, z) = r^2 + z^2$

And pay attention that there is a number

**Example:** Let $r = 2 \cos \theta$

So what happens is that projection onto $xy$ plane you integrate in polar coordinate system (so you find the limits in polar coordinate).
Example,

\[ r = 2 \cos \theta \text{ in } xy \text{ plane} \quad (\text{cylinder}) \]

and to the top \[ z = 9 - x \]

\[-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\]

\[ 0 \leq r \leq 2 \cos \theta \]

\[ 0 \leq z \leq 9 - r \cos \theta \]

\[ \iiint \, d\rho d\varphi dz \]

So first you write \[ \iiint \, d\rho d\varphi dz \], and find its limits. Then you write this and find its limits.
Spherical coordinate system

We might have

\[ (r, \theta, \phi) \]

instead of \((x, y, z)\)

we have \((r, \theta, \phi)\)

\[ \phi \geq 0 \]
\[ 0 \leq \theta \leq \pi \]
\[ 0 \leq \phi \leq 2\pi \]

\(r\) is a distance between point \(P\) and the origin. \(\theta\) is an angle (like in polar coordinates) and \(\phi\) is an angle between \(z\) axis and \(\overrightarrow{OP}\).
\[ \text{SSS } f(x, y, z) = p^2 \sin \theta \text{ d}p \text{ d}\theta \text{ d}\phi \]

Remember:
\[ x = p \cos \theta \sin \phi \]
\[ y = p \sin \theta \sin \phi \]
\[ z = p \cos \phi \]

So if you integrate a function \( f(x, y, z) \) then it becomes \( f(p \cos \theta \sin \phi, p \sin \theta \sin \phi, p \cos \phi) \).

1) For finding the values of \( \theta \) you rotate a plane (arrow-like plane).

First moment when does it touch and the last moment.
(Angle you measure between \( xz \) plane and this rotated plane (counterclockwise))
piece of ball $= 2$
of radius $= 2$

integrate $f(x,y,z)$ over this domain

$$\int \int \int_{V} f(x,y,z) \, dV$$

Find the limits of $\theta$

$0 \leq \theta \leq \frac{\pi}{2}$

$0 \leq \phi \leq \frac{\pi}{2}$

(first line it intersects and the last time $0 \leq \theta \leq \frac{\pi}{2}$

$0 \leq \phi \leq \pi$

Please review examples in the book