Double integrals

If you have a function of 1 variable, say \( f(x) \), then the question is to find the integral of this function over the given interval.

This means that you need to find endpoints of this interval, say \( a \) and \( b \), and compute the following integral:

\[
\int_{a}^{b} f(x) \, dx
\]
And the meaning of this expression \( \int_a^b f(x) \, dx \) is the area of the subgraph.

\[ \begin{array}{c}
\text{positive area} \\
\text{negative area}
\end{array} \]

Also note that the only natural sets on the real line are **intervals**.

The same question we can ask about the function of **two variables** \( f(x, y) \). But this function is defined on the **plane** (not on the real line).

So what would be natural sets
Over which we can integrate this functions?

Plane

natural sets rectangles

and this kind of sets.

So unlike the real line we have a lot of natural sets on the plane.

For simplicity let's consider only rectangles on the plane (and later we will
Cover other sets as well.

Take any rectangle. For example this one.

Then how would you define the integral of the function $f(x,y)$ over this rectangle (call it $R$)?

$$\int_{R} f(x,y) \, d\) of what? $d\times? \, dy?$

Something like this?

Here is the right way:

Since we have two variables, we write double integral.
\[ \iint_{R} f(x,y) \, dA \text{, this is not clear yet.} \]

OK, good.

In the case of one variable for the given interval we found endpoints of that interval. Now we have a rectangle and we want to find "endpoints" of that rectangle (or we would like to indicate that rectangle somehow).
How can we indicate the rectangle?

Well, I think this is the best way how can you indicate this rectangle. You see that variable $x$ varies between $a \leq x \leq b$ and variable $y$ varies between $c \leq y \leq d$. Then we write

$$\iint_R f(x,y) \, dA = \int_c^d \int_a^b f(x,y) \, dx \, dy$$
Note that:
\[ \int_{a}^{b} \int_{c}^{d} f(x,y) \, dx \, dy \]
\[ \int_{a}^{b} \int_{c}^{d} f(x,y) \, dy \, dx \]

But one can also write this integral in the following way (change the order):

\[ \iint_{R} f(x,y) \, dA = \int_{a}^{b} \int_{c}^{d} f(x,y) \, dy \, dx \]

Do you see the difference? We just changed the order.
Ok. How can one compute this kind of integrals?

$$\iint_{a,c} f(x,y) \, dy \, dx$$

First of all, remember that

$$\iint_{a,c} f(x,y) \, dy \, dx = \int_{c,a} \int_{a,c} f(x,y) \, dx \, dy$$

So if you integrate over the rectangle and you change the order then nothing changes!

Example:

Find $$\iint_{0,1} (x^2+y^2) \, dx \, dy$$
Solution:
\[ \int \int (x^2 + y^2) \, dx \, dy \]
for the region \( D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\} \)

\[ \text{1 step:} \int_{x=0}^{x=2} \left( \frac{x^3}{3} + y^2 \right) \, dx = \left( \frac{2^3}{3} + y^2 \right) - \left( \frac{1^3}{3} + y^2 \right) = \frac{8}{3} + y^2 \]

\[ \text{2 step:} \int_{y=0}^{y=1} \left( \frac{7}{3} + y^2 \right) \, dy = \left( \frac{7}{3} + \frac{y^3}{3} \right) \bigg|_{y=0}^{y=1} = \frac{7}{3} + \frac{1}{3} = \frac{8}{3} \]
Remark: if you change the order you get the same thing.

Indeed
\[ \int_0^2 \int_0^1 (x^2 + y^2) \, dy \, dx = \int_0^2 \left( x^2 y + \frac{y^3}{3} \right) \bigg|_{y=0}^{y=1} \, dx = \int_0^2 \left( \frac{x^3}{3} + \frac{x}{3} \right) \, dx = \frac{8}{3} + \frac{2}{3} - \left( \frac{1}{3} + \frac{1}{3} \right) = \frac{8}{3} \]

What about the integration over arbitrary set? y
You have to indicate this domain mathematically.

\[ a \leq x \leq b \quad \text{and for each fixed } x \]
\[ g(x) \leq y \leq f(x) \]

\[
\iint_{R} f(x,y) \, dA = \int_{a}^{b} \int_{g(x)}^{f(x)} f(x,y) \, dy \, dx
\]

Note that if you change the order
\[
\int_{a}^{b} \int_{g(x)}^{f(x)} f(x,y) \, dx \, dy
\]
you get nonsense (this is not a number)
But still can you change the order? Yes, but limits of integral will be different.

This corresponds to this order:

\[ \int_a^b \int_{g(x)}^{f(x)} F(x,y) \, dy \, dx \]

...and this picture:

\[ \int_c^d \int_{g^{-1}(y)}^{g^{-1}(y)} F(x,y) \, dx \, dy \]

...but it should contain...
which is wrong because it does not contain \( f(x) \). So you do the following:

\[
\iint_{D} F \, dA = \iint_{D_1} F \, dA + \iint_{D_2} F \, dA
\]

\[ \uparrow \text{additivity} \]

\[
\iint_{D_1} F \, dA = \int_{y_1}^{y_2} \left( \int_{x_1}^{x_2} F(x, y) \, dx \right) dy
\]

\[
\iint_{D_2} F \, dA = \int_{y_1}^{y_2} \left( \int_{x_1}^{x_2} F(x, y) \, dx \right) dy
\]
Remark: limits of integral define a domain and vice versa. The domain defines the limit of integrals.

Example:

Find \( \iint_D dA \) ?

Solution:

\[
\iint_D dA = \int_{0}^{2} \int_{x^2}^{2} dy\,dx = \int_{0}^{\frac{3}{2}} 2 - x^2\,dx = \left[ \frac{2x}{3} - \frac{x^3}{3} \right]_{0}^{\frac{3}{2}} = \frac{2\cdot\frac{3}{2}}{3} - \frac{2}{3} = \frac{1}{3}
\]
Let's see what do you get if you start computing in a different order

\[ 0 \leq x \leq \sqrt{2} \]
\[ 1 \leq y \leq 2 \]

but if \( x > 1 \),

then

\[ x^2 \leq y \leq 2 \]

So what to do?

Split domain in two parts.

\[ \iint dA = \]
\[ = \iint_{D_1} dA + \iint_{D_2} dA \]

in \( D_1 \), we have \( 0 \leq x \leq 1 \)
\[ 1 \leq y \leq 2 \]

in \( D_2 \) we have \( 1 \leq x \leq \sqrt{2} \)
\[ x^2 \leq y \leq 2 \]
Therefore you get:

\[ \iint_{D_2} dA = \int_0^{\sqrt{2}} y dy dx = \int_0^{\sqrt{2}} \frac{r_2^2}{r_2^2} \cdot \frac{r_2^2}{r_2^2} dx \]

\[ = 1 + \int_0^{\sqrt{2}} (2-x^2) dx = 1 + 2(\sqrt{2} - 1) - \frac{x^3}{3} \bigg|_1^{\sqrt{2}} \]

\[ = 2\sqrt{2} - 1 - \left( \frac{2\sqrt{2}}{3} - \frac{1}{3} \right) = \frac{4\sqrt{2}}{3} - \frac{2}{3} \]

the same answer

Example:

\[ \iint \ dy \ dx \] indicate the domain.

Solution:

\[ 1 \leq x \leq 2 \]

\[ 0 \leq y \leq x^3 \]

This is the answer.