Boundary Value Problems for Divergence Form Elliptic Equations

Steve Hofmann
University of Missouri, USA
hofmanns@missouri.edu

Abstract

The theory that we discuss in this lecture has its origin in the study of the steady-state heat equation, i.e., Laplace’s equation, which in $\mathbb{R}^d$ is

$$\Delta u := -\sum_{j=1}^d u_{x_j x_j} = 0.$$

Let $\Omega \subset \mathbb{R}^d$ be a connected open set (aka, a “domain”). The following pair of boundary value problems are fundamental: the Dirichlet Problem

$$\begin{cases}
\Delta u = 0 \text{ in } \Omega \\
\left. u \right|_{\partial \Omega} = f,
\end{cases} \quad (D)$$

and the Neumann Problem

$$\begin{cases}
\Delta u = 0 \text{ in } \Omega \\
\frac{\partial u}{\partial N} = g.
\end{cases} \quad (N)$$

In these problems, we seek to determine the temperature $u$ throughout $\Omega$, given either a specified temperature distribution $f$ on the boundary $\partial \Omega$ (Dirichlet problem), or the “heat flux” $g$ on the boundary (Neumann problem). Here, $\partial/\partial N$ denotes differentiation in the direction of $N$, the outer unit normal to the boundary. In practice, one typically specifies boundary data in some class, i.e., in some function space $X$, and one seeks appropriate estimates on the solution $u$, depending on the $X$-norm of the data.

In this talk, we shall review some of the classical theory, and then discuss more recent developments, in which the Laplacian $\Delta$ is replaced by more general elliptic operators.