True or false.

1. (2 Points) \( \int_{1}^{\infty} x^p \, dx \) converges if and only if \( p < -1 \).

True

2. (2 Points) \( \int_{0}^{1} x^p \, dx \) converges if and only if \( p > -1 \).

True

3. (2 points) \( \lim_{n \to \infty} a_n = 0 \) is enough to guarantee \( \sum a_n \) converges.

False. For example, the harmonic series diverges.

4. (3 points) Let \( f(n) \) and \( g(n) \) be any nonzero polynomials in \( n \), and let \( \rho > 0 \). Then the following three series all have the same ratio test value, \( \rho \):

\[
\sum_{n=1}^{\infty} f(n)\rho^n, \quad \sum_{n=1}^{\infty} \frac{f(n)\rho^n}{g(n)}, \quad \sum_{n=1}^{\infty} \frac{\rho^n}{g(n)}.
\]

True; polynomials are not detected by the ratio test.
5. (2 points) The three series in the above problem have the same convergence statuses.

**False.** If \( \rho = 1 \) and \( f(n) = n \) and \( g(n) = n^2 \), then the first series diverges by nTT, the second is (divergent) harmonic, and the third is a convergent p-series.

6. (2 points) Let \( k \geq 2 \) be an integer. The ratio test can be used to conclude that the following series converges

\[
\sum_{n=0}^{\infty} \frac{n!}{(n+k)!} = \frac{1}{k!} + \frac{1}{(k+1)!} + \frac{2}{(k+2)!} + \frac{3!}{(k+3)!} + \cdots
\]

**False**, the series converges, but the ratio test is inconclusive.

\[
\frac{n!}{(n+k)!} = \frac{1}{(n+k) \cdots (n+1)}.
\]

The denominator is a polynomial in \( n \) of degree \( k \), and polynomials are not detected by the ratio test.

7. (2 Points) Suppose \( a_n > 0 \) for all \( n \). Then

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} < 1 \implies \lim_{n \to \infty} a_n = 0.
\]

**True.** \( \rho < 1 \) implies convergence, which implies \( \lim a_n = 0 \). by nTT.

Note that the above limit is the ratio from the ratio test. And so, it being less than 1 is precisely the condition we need for convergence.

The \( n \)-th term test states that if \( \sum a_n < \infty \), then \( \lim a_n = 0 \).
(Equivalently, one often says that the \( n \)-th term test is the statement: \( \lim a_n \neq 0 \implies \sum a_n \) diverges.)
8. (2 points) The series $\sum_{n=1}^{\infty} \cos(n)/n$ is convergent, but not absolutely.

True.

9. (3 points) Let $x \in [0, \pi/2)$. Then $\sum_{n=0}^{\infty} (\sin x)^{2n} = \sec^2 x$.

True. Geometric series with $r = \sin^2 x$. 