1. (5+4+4+9+8=30 points) Determine the convergence status of the following infinite series. Please indicate when (and how) you are using a theorem or test.

(a) \[ \sum_{n=2}^{\infty} \frac{2n^{2n}(2n)!}{2n^{2n} - 2n} \]

Solution: The numerator of the summand is always significantly larger than the denominator. In particular, the summand \( a_n \) is always strictly greater than 1, and hence \( \lim a_n \neq 0 \). This series diverges.

(b) \[ \sum_{n=1}^{\infty} \frac{3}{\pi} \]

Solution: \( 3 < \pi \) so \( \rho = \left( \frac{3}{\pi} \right)^3 < 1 \). This is a convergent geometric series. Its sum is
\[
\frac{\left( \frac{3}{\pi} \right)^3}{1 - \left( \frac{3}{\pi} \right)^3} = \frac{9}{\pi^3 - 9}.
\]

(c) \[ \sum_{n=2}^{\infty} (4 - e)^{2n+1} \]

Solution: \( e < 3 \) so \( \rho = (4 - e)^2 > 1 \). This is a divergent geometric series.

(d) \[ \sum_{n=1}^{\infty} \frac{n!}{n^n} \]

Solution: This series converges by the ratio test:
\[
\rho = \lim \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} = \lim \left( \frac{n}{n+1} \right)^n = \lim \left( 1 + \frac{1}{n} \right)^{-n} = e^{-1} < 1
\]
The important thing to note here is that \( \arctan \) is a bounded function.

\[ 0 \leq \arctan n < \frac{\pi}{2} \quad \text{for} \quad n \in \mathbb{N}. \]

Therefore

\[
\sum_{n=1}^{\infty} \frac{2n^2 \arctan n}{\pi n^\pi + n} \leq \sum_{n=1}^{\infty} \frac{2n^2 \arctan n}{\pi n^\pi} < \sum_{n=1}^{\infty} \frac{n^2}{n^\pi}
\]

which is a convergent \( p \)-series with \( p = \pi - 2 > 1 \). Our series converges by direct comparison.

2. (5+6+7=18) Give the interval of convergence of the following power series. You do need to check the endpoints.

(a)

\[
\sum_{n=1}^{\infty} \frac{(x - 2)^n}{(2n)!2^n}
\]

**Solution:** This series converges for all \( x \in \mathbb{R} \) by the ratio test:

\[
\rho = \lim_{n \to \infty} \left| \frac{x - 2}{2(2n + 2)(2n + 1)} \right| = 0 < 1.
\]

(b)

\[
\sum_{n=1}^{\infty} \frac{(x - 2)^{3n}}{\sqrt{n}8^n}
\]

**Solution:**

\[
\rho = \lim_{n \to \infty} \sqrt[n]{\frac{n}{n + 1}} \left| \frac{x - 2}{2} \right|^3 = \left| \frac{x - 2}{2} \right|^3.
\]

This is less than 1 when \( x \in (0, 4) \). The left endpoint gives a convergent alternating series, whereas the right gives a divergent \( p \)-series with \( p = 1/2 \). So the interval of convergence is \( [0, 4) \).
(c) \[ \sum_{n=7}^{\infty} \frac{x^n}{n(\ln n)^2} \]

Solution:

\[ \rho = \lim_{n \to \infty} \left( \frac{n}{n+1} \left( \frac{\ln n}{\ln(n+1)} \right)^2 |x| \right) = |x|. \]

This is less than 1 when \( x \in (-1, 1) \). The right endpoint is convergent by the integral test

\[ \int_{1/2}^{\infty} \frac{dx}{x(\ln x)^2} = -\frac{1}{\ln x} \bigg|_{1/2}^{\infty} = \frac{1}{\ln 2} < \infty, \]

and the left endpoint is the alternation of the right, and thus also converges. The interval of convergence is \([-1, 1]\).

3. (6+9=15 points)

(i) Give a 4th order polynomial approximation of \( \ln \left( \frac{11}{10} \right) \).

(ii) Use the remainder estimation theorem to give an estimate on the maximum error of the above approximation.

Hint:

\[ \ln x = \int_{1}^{x} \frac{dt}{t} = \int_{1}^{x} \frac{dt}{1 - (1 - t)} \]

and \( \sum_{n=0}^{\infty} x^n = \frac{1}{1 - x} \)

Solution: From the hint we have

\[ \ln x = \int_{1}^{x} \left( \sum_{n=0}^{\infty} (1 - t)^n \right) dt = \sum_{n=0}^{\infty} \left( \int_{1}^{x} (-1)^n (t - 1)^n dt \right) = \]

\[ \sum_{n=0}^{\infty} \frac{(-1)^n (x - 1)^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x - 1)^n}{n} \]

Therefore

\[ \ln \left( \frac{11}{10} \right) \approx P_4 \left( \frac{11}{10} \right) = \sum_{n=1}^{4} \frac{(-1)^{n+1} \left( \frac{11}{10} - 1 \right)^n}{n} = \frac{1}{10} - \frac{1}{200} + \frac{1}{3000} - \frac{1}{40000}. \]
(ii) Now to estimate the error of the approximation we note the \( \ln x \) has \(|\ln^{(k)}(x)| = (k - 1)!x^{-k} \leq (k - 1)! \) on \( x \in [1, 11/10] \). We have the following bound on the error:

\[
|R_4(11/10)| \leq \frac{(5 - 1)!\left(\frac{11}{10} - 1\right)^5}{5!} = \frac{1}{500000}.
\]

This number is exactly the magnitude of the next term of the series. This is what we would expect, since the series is alternating.

4. (12 points) Write a 4th order Taylor polynomial with remainder for \( f(x) = 2^x \) centered at \( a = 1 \). (Hint: remember, \( 2^x = e^{x\ln 2} \).)

**Solution:** We make the observation \( f^{(k)}(x) = (\ln 2)^k f(x) \). Then \( f^{(k)}(1) = (\ln 2)^k 2 \) and we have

\[
f(x) = \sum_{k=0}^{4} \frac{f^{(k)}(1)}{k!} (x - 1)^k + \frac{f^{(5)}(c)}{5!} (x - 1)^5 = \]

\[
2 + 2(\ln 2)(x - 1) + 2\frac{(\ln 2)^2(x - 1)^2}{2!} + 2\frac{(\ln 2)^3(x - 1)^3}{3!} + \]

\[
+ 2\frac{(\ln 2)^4(x - 1)^4}{4!} + 2\frac{(\ln 2)^5c(x - 1)^5}{5!}
\]

for some \( c \) between \( x \) and 1 which is guaranteed to exist by Taylor’s theorem.

5. (7+7=14 points) Find a power series representation for the following functions

(a) \( f(x) = x \arctan x \)

**Solution:** Starting with a geometric series with ratio \( -x^2 \) we obtain

\[
\frac{d}{dx} \arctan x = \frac{1}{1 + x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}
\]

which we integrate and multiply by \( x \) :

\[
x \arctan x = x \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{2k+1}
\]
(b)

\[ g(x) = \left( \frac{x}{1-2x} \right)^2 \]

**Solution:** First make the observation

\[ \frac{d}{dx} \frac{1}{1-2x} = \frac{2}{(1-2x)^2}. \]

We can then write

\[ g(x) = \frac{x^2}{2} \frac{d}{dx} \sum_{n=0}^{\infty} 2^n x^n = \frac{x^2}{2} \sum_{n=1}^{\infty} n2^n x^{n-1} = \sum_{n=1}^{\infty} n2^{n-1} x^{n+1} \]

6. (11 points) The Lorentz Factor is a term which appears in several of the equations from special relativity. It is defined as

\[ \gamma = \frac{1}{\sqrt{1-\beta^2}} \]

where \( \beta = v/c \) is the ratio of the speed, \( v \), of a given reference frame, to the speed of light, \( c \).

Give a power series representation of \( \gamma \) in terms of \( \beta \).

**Solution:** \( \gamma = (1 + x)^p \) is binomial with \( p = -1/2 \) and \( x = -\beta^2 \)

\[ \gamma = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n \beta^{2n} = 1 + \frac{\beta^2}{2} + \frac{3\beta^4}{8} + \cdots \]