What (Future) High School Math Teachers Need to Know about Trigonometry

by

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Future high school math teachers almost always see trigonometry in high school, perhaps again in a college precalculus course, if they take one. They see the small amount in Calculus 1 & 2 that is necessary to compute some derivatives and integrals. After that many do not see any more trig for the rest of their undergraduate career, except possibly ODEs and Fourier series. In a capstone course for future high school teachers we have been developing, most of the students had forgotten much of trig and had questions about what they partially remembered. Thus, if they had to begin to prepare to teach a lesson on trig, they would have little in their memory to fall back upon. Anecdotal evidence indicates that many in-service high school math teachers are in a similar situation. Thus we have developed the following outline that sketches the basic facts and relationships that we feel high school teachers should have at their fingertips about trigonometry. We see this as an aid to (future) teachers to fill in their own particular gaps.[2] Some detailed mathematical and pedagogical comments are given after the outline. In current education parlance, the first section is mostly content knowledge and the second section is mostly pedagogical content knowledge.

1. In geometry, an angle consists of two rays with a common vertex. In trigonometry, an angle is this together with the amount and direction of a rotation that takes one ray to the other.

2. The sum of the angles in a triangle is 180 degrees. Similar triangles have equal angles and corresponding sides are proportional.

3. Pythagorean Theorem. Know at least two proofs.

4. The definition of the six trigonometric functions of an acute angle θ in terms of ratios of lengths of sides of a right triangle with θ as one angle. Explain why they are well-defined.

5. Basic identities (for acute angles at this point):

a. cos θ is the sine of the *co*mplementary angle of θ. (Two more similar ones.)

b. tan θ = sin θ/cos θ, cot similarly.

c. Three Pythagorean identities (and their equivalent forms such as $\cos^2\theta = 1-\sin^2\theta$).

d. The area of a triangle is given by $A=\frac{1}{2}ab\sin\theta$ (where *a* and *b* are the lengths of two sides and θ is the included angle).

6. Know the lengths of the sides of a 45-45 and a 30-60 right triangle. Be able to derive these lengths and to use them to find the trig functions of the special angles.

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[1] With much input from Dick Askey, U. of Wisconsin.

[2] It is assumed that anyone reading this list will have some background in trigonometry.

7. Generalize the trig functions to functions of general angles or real numbers. (Teachers should know at least three ways to do this.)

8. The basic identities in 5 for the general situation. (These are almost immediate from the definitions.)

9. Evaluate the trig functions of the general special angles by scaffolding on 6.

10. Derive the graphs of sin, cos, tan and their inverses. Derive the basic properties of the inverse trig functions from their graphs.

11. Be able to prove the addition formulas for sin and cos, first geometrically when θ is in a triangle then algebraically for general angles. Then starting with these, be able to derive the flow first through the double-angle and then the half-angle formulas.

12. Laws of Sines and Cosines. Be able to derive and apply.

13. Be able to do all the various applications that arise along the way.

MATHEMATICAL AND PEDAGOGICAL COMMENTS

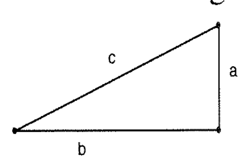
The approach used here is the right-triangle approach. Another approach is to start in with the unit circle. This approach works, and the people who like it feel it helps students understand trig functions as functions better. However, the right-triangle approach mimics the way trigonometry evolved, and it seems to lend itself to scaffolding better. Also, the unit circle approach hides similarity (which will be needed for applications), and although the sine and cosine functions are very natural on the unit circle, the other trig functions are not.

1. In coordinate trigonometry, that angle is almost always in standard position.

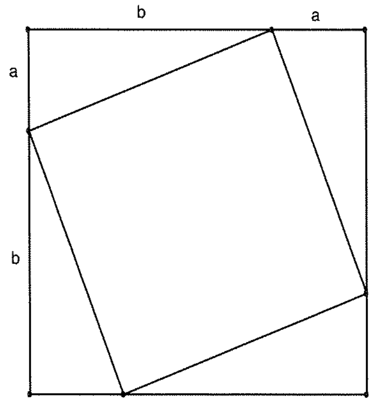
2. One definition of similar polygons is that (a) their corresponding angles are equal and (b) their corresponding sides are proportional. It is then a theorem that for triangles (a) and (b) imply each other. This takes a surprising amount of care to prove. Another definition of similarity uses expansion functions of the plane. This approach seems a little easier except that it requires more sophistication of the students, which some do not have.

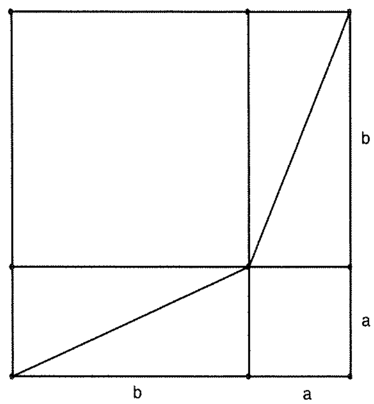
3. Google "Pythagorean Theorem" for numerous proofs. The author's preference (which is completely geometric and sixth graders can follow) is:

Start with a right triangle with sides of length a, b, c:

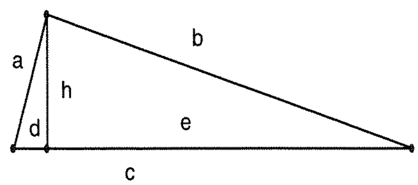


Next form the following two squares of equal area and discern the proof (after first arguing the inner quadrilateral on the left is a square):





For a second proof with a little interesting algebra, from the vertex of the right angle drop a perpendicular to the opposite side:



Here, $c = d + e$. Show that the resulting three triangles are similar. Since ratios of corresponding sides are equal, we obtain $\frac{d}{a} = \frac{a}{c}$, $\frac{e}{b} = \frac{b}{c}$, so that $a^2 + b^2 = cd + ce = c^2$.

5. From geometry they should know how to derive $A = \frac{1}{2}bh$ from the area of a rectangle and start here for part d.

7. Some of the ways are: using a unit circle; using a circle of radius $r > 0$, wrapping functions, and triangles in different quadrants (using "directed distances" for the legs). This last way was used to find reference angles and the value of trig functions from tables. It has fallen out of favor now that we have calculators.

9. It has become a pretty well-understood principal of teaching that scaffolding, i.e., building new knowledge on top of old, should be used as much as possible. This can be done with evaluating trig functions of general special angles if a student knows #6, above. Some trig teachers have their students memorize the coordinates of all of the special angles on the unit circle. This practice is hard on average and weaker students and ignores scaffolding.

10. Here it might be mentioned that sin and tan are odd functions and cos is even; the consequences should be related to their graphs. Once the graphs are derived and

explained, there is an easy way for students to remember the values of sin and cos (and tan) at multiples of $\frac{\pi}{2}$.

a. Draw the graph on a set of axes *without* labeling points first.
b. Lable ± 1 on the y-axis and 0, 2π on the x-axis; read off the values at 0, 2π.
c. Lable π on the x-axis; read off the value at π.
d. Lable $\pi/2, 3\pi/2$ on the x-axis; read off the values there.

After doing this several times, students will learn to do this all in their heads.

11. The formulas $\sin^2 x = \frac{1-\cos 2x}{2}$, $\cos^2 x = \frac{1+\cos 2x}{2}$ are equivalent to the half-angle formulas but are more useful in calculus.

The addition formula for tan is useful for finding angles between lines, and some people advocate that students, and especially teachers, should know how to derive it from the addition formulas for sin and cos.

It cannot be overemphasized that having students learning the flow

sum formulas -> double-angle formulas -> half-angle formulas,

not memorizing the formulas separately, helps students' understanding (because of the principal of scaffolding).

There are other identities found in trig books (such as sin u sin v = …) which are useful in calculus and higher mathematics. Some are pretty neat. The connections to complex numbers perhaps should be covered, too. However, the ones indicated here are basic, and the author would be delighted if these (and their derivations) were generally known.

Finally, for those interested in a supplemental book, please look at Eli Mayor, *Trigonometric Delights*, Princeton University Press, 1998. This is a wonderful little book full of historical perspectives, interesting applications, and various trigonometric relationships. There is much to delight math teachers and their students.