Future High School Math Teachers and Upper Level Math Courses[1]

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There is considerable anecdotal evidence that high school math teachers do not see the relevance of the upper level math courses they took in college to the mathematics that they are teaching in high school. Two programs at Michigan State University have led several of us to look at this situation more closely. The first is a Teachers for a New Era (TNE) grant whose purpose is to examine our undergraduate education of future teachers very closely and identify improvements that might be made. The second is the development of a new senior-level capstone course for (math majors who are) future secondary math teachers that is jointly taught by a mathematician and a math educator.

We have discovered that our upper level math courses often miss opportunities to draw connections with high school math. Sometimes the relevant high school math is considered too "elementary" to be mentioned in the college course. Sometimes the topic is mentioned, but not sufficiently emphasized nor connected to the high school math, and the students forget it by time they take the capstone course (and hence would have forgotten it by the time they got ready to teach it in high school). Finally some topics, such as trigonometry, have been at best briefly surveyed in post-precalculus courses. These and related issues are discussed below. Given how important it is for us to educate future high school teachers well, perhaps mathematics departments should encourage instructors in upper-level undergraduate math courses to including discussions about such connections.

Here is the list of topics and courses in which they are found together with comments.

1. Complex analysis: $\sqrt{-4}\sqrt{-9} \neq 6$. This is an example of something usually considered "too elementary" to appear in a complex analysis course. But perhaps it should, since most of our future high school teachers had not seen it before, and they were somewhat skeptical since they had grown up with $\sqrt{a}\sqrt{b} = \sqrt{ab}$. The "$\neq$" for complex numbers follows from the usual convention of picking which of the two square roots to be the principal square root. The explanation is likely to involve discussing $f(z) = z^2$ as a function of the unit circle to itself that wraps the circle around itself twice. Most have not seen a description of a function like that before.
2. Real analysis: How to add and multiply infinite decimals and why 0.999… is equal to 1.000… Of course, these topics can be handled using power series. However, they are usually not mentioned more than superficially in either calculus or real analysis. Senior math majors in our capstone course generally

[1] An abridged version of this article appears in the Notices of the AMS, Dec. 2005, p.1317.

were not even aware there might be a question about how to add the decimal expansions of 5/7 + 8/9 (without adding the fractions first). As future math teachers they will be dealing with infinite decimals, and they should know this topic well enough to explain the issues and their solutions to students who might ask about them.

3. Lines and planes in 3- space are topics in multivariable calculus, and elementary versions are topics of high school mathematics. Each is determined by a point and a vector. In both cases the point is on the object; the vector is parallel to the line or perpendicular to the plane. Lines and hyperplanes are subspaces of *n*-space, so they are topics of linear algebra. Their descriptions generalize the descriptions of lines and planes in 3-space. Perhaps these should have been emphasized more, because our capstone students had forgotten essentially all of this. We designed a project for them of taking the equations for lines and planes in 3-space and restricting them to 2-space. This yields two descriptions of lines in 2-space, which the students were asked to reconcile. This turned out to be a very challenging project, but one that is sure to be valuable for them when they teach.
4. Linear algebra has two more topics relevant for high school mathematics. It would be valuable to the future teachers for instructors to point this out and not to underemphasize them.
    a. Matrices that induce rotations and reflections in the plane (and maybe in 3-space) have become a common topic in high school precalculus courses.
    b. The least-squares problem, i.e., fitting a regression line to a collection of points in a plane. In high school, these are found by pushing the appropriate buttons on a calculator; no theory is necessary. But teachers can (and sometimes do) prove why this works using properties of parabolas. But teachers will understand the background better using the elegant linear algebraic arguments.
5. Abstract algebra has several topics that are important for future teachers, and hence they need to be emphasized by the instructors of the course. The first three of these future teachers themselves might easily be teaching in Algebra 2 and Precalculus.
    a. The Division Algorithm. They need to know this cold, both for *Z* and for polynomials over a field, *F*. In particular they need to know why all of the hypotheses are necessary.
    b. The flow of theorems: Division algorithm -> Remainder Theorem -> Factor Theorem.[2]
    c. The confluence of b. together with the Fundamental Theorem of Algebra (which we prove in the capstone course) to describe factoring of polynomials over C into linear factors and over R into linear and quadratic polynomials.
    d. Use the quotient $R[x]/(x^2 + 1)$ to explain why it is rigorous to describe addition and multiplication in C as "just like polynomials except that $x^2 = -1$."

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[2]Remainder Theorem: For p in $F[x]$, a in F, and $p(x) = (x - a)q(x) + r$, then $p(a) = r$.
Factor Theorem: For p in $F[x]$, a in F, then $x - a$ is a factor of p if and only if $p(a) = 0$.

6. Trigonometry. To our surprise, most of the students had very little trig at their fingertips (which we discovered when we started "reviewing" the polar form of complex numbers). They had not seen much trig since high school, except for graphs and a few identities needed in calculus. They did not realize an angle is defined differently in geometry and in trigonometry[3], they had not thought about different definitions of the trig functions[4], they did not know why the formula for sin(u + v) is important[5], and in general they did not know many identities. As an undergraduate, I had a two-week review crash course on trig in calc. 2 before starting integration formulas, but there is no longer room for this since series was moved to calc. 2. The second time we taught the capstone course, we gave these students a thorough review, emphasizing things they will need to understand when they teach. But is this something the mathematical community should worry about for all math majors?

6. General comments.
    a. Functions. Essentially these students knew a function has inputs and outputs such that every input has a unique output. Several never had to memorize a formal definition, such as "A function consists of two sets and a rule ..." or "A function is a set of ordered pairs ..." This means that students going off to graduate school may not have either, and I have concerns about that. In any case, future high school teachers need to know all three definitions and understand their equivalence, as all three appear in high school math textbooks. Should they get this earlier than a capstone course and if so where?
    b. Graphs. Except for functions from the reals to the reals, students have no idea where the graph of a function lies. They can get very creative in discussing graphs of rational functions that have factors like $x^2 + 4$ in the denominator and get confused about where the asymptotes at +/- $2i$ go. Is this something else we should worry about for all math majors?
    c. Theory. Generally, students should understanding why precise definitions are important and have more experience constructing counterexamples, especially when hypotheses in theorems are deleted. Our future math teachers (and hence our math majors in general) saw very little of this, but are not such things a central part of understanding mathematics?

Undoubtedly many people reading this will come up with other concerns both for our future high school math teachers and for our math majors in general. It seems like a crucial time for mathematics departments to discuss these issues. An old Chinese proverb says the best time to plant a tree to shade your yard was twenty years ago; the second best time is now.

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[3] In geometry it is two rays emanating from a common point; in trig, it is this plus an amount of rotation taking one ray to the other.
[4] I.e., angles of right triangles vs generalized angles vs coordinates of points on the unit circle.
[5] Among other things, for deriving formulas for the derivative of sin and cos, and for deriving identities useful in calc. 2.

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