3. General Properties of Differential Equations

Let $\mathbb{R}^{n+1}$ be the $(n+1)$-dimensional Euclidean space and let $(t,x)$ denote coordinates in $\mathbb{R}^{n+1}$ with $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Write $\dot{x} = \frac{dx}{dt}$.

A first order ordinary differential equation in $\mathbb{R}^n$ is an expression of the form

\begin{equation}
\dot{x} = f(t,x),
\end{equation}

where $f$ is a function from an open set $D \subseteq \mathbb{R}^{n+1}$ to $\mathbb{R}^n$. When $f$ depends explicitly on $t$, the equation (3.1) is called nonautonomous or time dependent. If $f$ is independent of $t$, it is called autonomous or time independent.

A solution to (3.1) is a differentiable function $x(t)$ from a real interval $I$ into $\mathbb{R}^n$ so that

1. $\{ (t, x(t)) : t \in I \} \subseteq D$,
2. For $t \in I$, $\dot{x}(t) = f(t,x(t))$.

If we fix a point $(t_0,x_0) \in D$, we are sometimes interested in solutions $x(\cdot)$ of (3.1) for which $x(t_0) = x_0$.

This leads us to the system of equations

\begin{equation}
\dot{x} = f(t,x), \quad x(t_0) = x_0,
\end{equation}

which we will call the initial value problem of the differential equation (3.1) with initial value $(t_0,x_0)$, or simply the initial value problem.

Remarks.

1. The $n$-th order scalar differential equation

\[
\frac{d^n x}{dt^n} = g(t,x,\dot{x}, \frac{d^2 x}{dt^2}, \ldots, \frac{d^{n-1} x}{dt^{n-1}})
\]

can be written as the vector system

\[
\begin{align*}
\dot{x} &= x_1 \\
\frac{dx_1}{dt} &= x_2 \\
& \vdots \\
\frac{dx_{n-1}}{dt} &= g(t,x_1,\ldots,x_{n-1})
\end{align*}
\]

using the vector, $(t,x,x_1,\ldots,x_{n-1})$ with $x_i = \frac{d^i x}{dt^i}$ so it is usually not necessary to consider higher order differential equations for general properties.
(2) In issues in which \( f(t, x) \) is very smooth, e.g. \( C^\infty \), it is frequently useful to replace the non-autonomous equation (3.1) by the system \( \dot{t} = 1 \), \( \dot{x} = f(t, x) \) and obtain an autonomous equation in one higher dimension.

**Examples.**

(1) The first example shows that even if the right hand side of a differential equation is a polynomial, solutions to (3.1) may not be defined for all real time.

Let \( D = \mathbb{R}^2 \), \( f(t, x) = x^2 \). The initial value problem

\[
\dot{x} = x^2, \quad x(0) = x_0
\]

has the unique solution \( \phi(t) = -\frac{1}{t - x_0^{-1}} \) for \( x_0 \neq 0 \) and \( \phi(t) = 0 \ \forall t \)

for \( x_0 = 0 \). For \( x_0 \neq 0 \), these solutions blow up in finite time.

(2) The second example shows that the initial value problem of a continuous differential equation need not have a unique solution.

Let \( D = \mathbb{R}^2 \),

\[
f(t, x) = \begin{cases} \sqrt{x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}
\]

Fix a real number \( c > 0 \), and define the function

\[
\phi_c(t) = \begin{cases} \frac{(t-c)^2}{4} & \text{for } t \geq c, \\ 0 & \text{for } t < c. \end{cases}
\]

Then, each \( \phi_c(t) \) is a solution to \( \dot{x} = f(t, x) \) with value 0 at \( t_0 = 0 \).

**Lemma 3.1.** Suppose that \( f(t, x) \) is a continuous function on an open set \( D \) in \( \mathbb{R}^{n+1} \). Let \((t_0, x_0) \in D\). Then, a continuous function \( x(t) \) is a solution to the single integral equation

\[
(3.3) \quad x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) \, ds.
\]

if and only if it is a solution to the initial value problem (3.2).

**Proof.** “\( \Rightarrow \)” Suppose that \( x(\cdot) \) is a continuous function which solves the integral equation. Then, \( x(t_0) = x_0 \), and since \( f \) is continuous, the Fundamental Theorem of Calculus gives that \( x(t) \) is differentiable with \( \dot{x} = f(t, x(t)) \) so that \( x(\cdot) \) solves (3.2).

“\( \Leftarrow \)” Conversely, suppose that the \( x(\cdot) \) is a solution to the problem (3.2). Then, \( x(\cdot) \) is differentiable, hence continuous, on an interval about \( t_0 \). Let \( h(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) \, ds \).
Again the Fundamental Theorem of Calculus gives that \( h \) is differentiable with derivative \( f(t, x(t)) \) at \( t \). Thus, both \( x(t) \) and \( h(t) \) are differentiable functions with the same derivative on an interval about \( t_0 \). Hence, they differ by a constant. But they both have the value \( x_0 \) at \( t_0 \), so the constant is 0, and \( x(t) \) solves the integral equation.

We wish to show that differential equations with continuous right hand sides have solutions at least on small intervals.

**Theorem 3.2** (Peano Existence Theorem). Suppose that \( f(t, x) \) is continuous in the open set \( D \subseteq \mathbb{R}^1 \times \mathbb{R}^n \). Then, for \((t_0, x_0)\) in \( D \), the initial value problem (3.2) has at least one solution.

We will give two proofs of this theorem. The first depends on a theorem in Functional Analysis.

**Proof 1 of Peano Theorem.** For \( \alpha > 0, \beta > 0 \) let
\[
I_\alpha = I_\alpha(t_0) = \{ t : |t - t_0| \leq \alpha \},
B_\beta = B_\beta(x_0) = \{ x : |x - x_0| \leq \beta \}.
\]
Choose \( \alpha, \beta \) small enough so that \( I_\alpha \times B_\beta \subseteq D \).
Since \( I_\alpha \times B_\beta \) is compact and \( f \) is continuous on \( I_\alpha \times B_\beta \), the quantity
\[
M = \sup \{|f(t, x)| : (t, x) \in I_\alpha \times B_\beta \}
\]
is finite.
Let \( \alpha_1 \) be positive and small enough so that \( M\alpha_1 \leq \beta \).
Let
\[
\mathcal{A} = \{ \phi \in C(I_{\alpha_1}, \mathbb{R}^n) : \phi(t_0) = x_0, |\phi(t) - x_0| \leq \beta \ \forall t \in I_{\alpha_1} \}.
\]
Clearly \( \mathcal{A} \) is a closed bounded convex subset of the Banach space \( C(I_{\alpha_1}, \mathbb{R}^n) \) with the sup norm. Let \( T : C(I_{\alpha_1}, \mathbb{R}^n) \to C(I_{\alpha_1}, \mathbb{R}^n) \) be defined by
\[
(T\phi)(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s)) \, ds.
\]
By the claims below, the Extended Schauder Fixed Point Theorem gives us a fixed point \( \psi \) of \( T \) in \( \mathcal{A} \). This fixed point solves the integral equation (3.3), so it provides a solution to the IVP (3.2).

**Claim 1.** \( T \) maps \( \mathcal{A} \) into itself.

**Claim 2.** \( T \) is continuous.

**Claim 3.** \( T \mathcal{A} \) has compact closure.
Proof of Claim 1. Let $\phi \in \mathcal{A}$. Clearly, $I_{\alpha_1} \times \phi(I_{\alpha_1}) \subseteq D$ so $T$ is well-defined. Also, $(T\phi)(t_0) = x_0$. Next, for $t \in I_{\alpha_1}$,
\[ |(T\phi)(t) - x_0| \leq \left| \int_{t_0}^{t} f(s, \phi(s))ds \right| \leq M\alpha_1 \leq \beta. \]
Hence, $T\phi \in \mathcal{A}$. $\square$

Proof of Claim 2. Let $\epsilon > 0$. We know that $f$ is uniformly continuous on $I_{\alpha_1} \times B_\beta$. Take $\delta > 0$ such that if $|(t, x) - (s, y)| < \delta$ and $(t, x), (s, y) \in I_{\alpha_1} \times B_\beta$, then $|f(t, x) - f(s, y)| < \epsilon/\alpha_1$.

Suppose that $\phi, \psi \in \mathcal{A}$ are such that $\|\phi - \psi\| < \delta$. This means that, for each $t \in I_{\alpha_1}$, $|\phi(t) - \psi(t)| < \delta$. Thus, for $t \in I_{\alpha_1}$,
\[ |(T\phi)(t) - (T\psi)(t)| \leq \left| \int_{t_0}^{t} f(s, \phi(s)) - f(s, \psi(s))ds \right| \leq (\epsilon/\alpha_1)|t - t_0| \leq (\epsilon/\alpha_1) \cdot \alpha_1 = \epsilon. \]
Hence, $\|T\phi - T\psi\| \leq \epsilon$. So $T$ is continuous on $\mathcal{A}$. $\square$

Proof of Claim 3. First, note that $TA$ is equicontinuous. In fact, for any $\phi \in \mathcal{A}, t, u \in I_{\alpha_1}$,
\[ |(T\phi)(t) - (T\phi)(u)| \leq \left| \int_{u}^{t} f(s, \phi(s))ds \right| \leq M|t - u|. \]
It follows that the closure of $TA$ is also equicontinuous. Since it is also bounded, it will follow from the Arzela-Ascoli Theorem that $TA$ has compact closure as required. $\square$

Proof 2 of Peano Theorem. Let $I_{\alpha}, I_{\alpha_1}, B_\beta$ be as in Proof 1.

Take $n \geq 1$, and let $h = h_n = \frac{\alpha_1}{n}$.

We will consider the Euler polygonal approximations $\phi_h$ for solutions defined in the following way.

First, let $t_1 = t_0 + h$ and $x_1 = x_0 + f(t_0, x_0)h$. Then for $1 \leq i \leq n - 1$, let
\[ t_{i+1} = t_i + h = t_0 + ih, \quad x_{i+1} = x_i + f(t_i, x_i)h. \]
This is a discrete sequence of vectors. Interpolate linearly between $(t_i, x_i)$ and $(t_{i+1}, x_{i+1})$ to form the function
\[ \phi_h(t) = x_i + f(t_i, x_i)(t - t_i) \quad \text{for } t_i \leq t \leq t_{i+1}. \]
By Claim 1 below, all $x_i$ are in $B_\beta$ and hence $|f(t_i, x_i)| \leq M$. By definition, $\phi_h$ is a linear function on each interval $[t_i, t_{i+1}]$ for $i = 0, \ldots, n-1$ with slope $f(t_i, x_i)$. So $\phi_h$ is a Lipschitz function with Lipschitz constant less than or equal to $M$. Since this is true for any $n$, the sequence
\{ \phi_{h_n} \} is equicontinuous. Also, the fact that \( x_i \in B_\beta \) and \( \phi_{h_n} \) is piecewise linear for any \( n \) implies that \( \{ \phi_{h_n} \} \) are uniformly bounded.

Thus, by the Arzela-Ascoli theorem, there is a sequence \( \phi_{h_{n_i}} \) which converges to a function \( \psi \) defined on \( I_{\alpha_1} \). Claim 2 below gives that

\[
\left| \psi(t) - x_0 - \int_{t_0}^t f(s, \psi(s)) \, ds \right| = 0,
\]
or equivalently,

\[
\psi(t) = x_0 + \int_{t_0}^t f(s, \psi(s)) \, ds.
\]

That is, \( \psi \) is a solution of (3.2).

\[ \Box \]

Claim 1. \( |x_i - x_0| \leq \beta \) for any \( i = 0, 1, \ldots, n \).

Claim 2. As \( h_n \to 0 \), \( \left| \phi_{h_n}(t) - x_0 - \int_{t_0}^t f(s, \phi_{h_n}(s)) \, ds \right| \to 0 \).

**Proof of Claim 1.** We prove it by induction.

Clearly it is true for \( i = 0 \).

Suppose \( |x_j - x_0| \leq \beta \) for all \( j = 0, 1, \ldots, i \). Then \( (t_j, x_j) \in I_{\alpha_1} \times B_\beta \) and therefore

\[
|f(t_j, x_j)| \leq M.
\]

So

\[
|x_{i+1} - x_0| = |x_i + f(t_i, x_i) h - x_0|
\]

\[
= |x_0 + \sum_{j=0}^i f(t_j, x_j) h - x_0|
\]

\[
\leq M \cdot (i + 1) h \leq M \cdot \alpha_1 \leq \beta,
\]

whenever \( i < n \). Now the claim follows from induction.

\[ \Box \]

**Proof of Claim 2.** We leave it as an exercise.