16. Hamiltonian Systems in $\mathbb{R}^{2n}$

Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a $C^k$ function, $k \geq 1$. Write coordinates $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ on $\mathbb{R}^{2n}$.

A system of differential equations of the form

$$
\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \ldots, n
$$

is called a Hamiltonian system with $n$ degrees of freedom and Hamiltonian function $H$. We also write $X_H$ for the vector field defined by (16.1).

Sometimes we write the shortened form of (16.1) as

$$
\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q},
$$

where $q = (q_1, \ldots, q_n), p = (p_1, \ldots, p_n)$. If we define $z = (q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$,

$$
\nabla H = \left( \frac{\partial H}{\partial q_1}, \ldots, \frac{\partial H}{\partial q_n}, \frac{\partial H}{\partial p_1}, \ldots, \frac{\partial H}{\partial p_n} \right)
$$

and

$$
J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},
$$

where $I$ is the $n \times n$ identity matrix, then (16.1) has the form

$$
\dot{z} = J \nabla H (z).
$$

The matrix $J$ above is called the standard symplectic matrix. It is one of the normal forms of a non-degenerate alternating bilinear form on $\mathbb{R}^{2n}$. Because of equation (16.2), one sometimes refers to a Hamiltonian system as a symplectic gradient. However, the orbit structure of a Hamiltonian system is vastly different from that of a gradient system.

**Proposition 16.1.** If $X_H$ is a Hamiltonian system with Hamiltonian $H$, then $H$ is constant on orbits.

**Proof.** For any solution curve $\gamma(t) = (q(t), p(t))$ we have

$$
\frac{dH(q(t), p(t))}{dt} = \sum_{i=1}^{n} \frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p_i} \dot{p}_i
$$

$$
= \sum_{i=1}^{n} \frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial p_i} \left( -\frac{\partial H}{\partial q_i} \right) = 0.
$$

**Definition 16.1.** A $C^1$ (nonconstant) function $\Phi : \mathbb{R}^n \to \mathbb{R}$ is a first integral for a differential equation $\dot{x} = f(x)$ defined on $\mathbb{R}^n$ if $\Phi$ is constant on any orbits.
The above proposition says that the Hamiltonian function is a first integral of the corresponding Hamiltonian system.

**Definition 16.2.** A map \( \phi : \mathbb{R}^n \to \mathbb{R}^n \) is volume preserving if for any measurable set \( E \subset \mathbb{R}^n \), \( \phi^{-1}(E) \) and \( E \) have the same Lebesgue measure.

**Proposition 16.2.** A \( C^1 \) diffeomorphism \( \phi : \mathbb{R}^n \to \mathbb{R}^n \) is volume preserving if and only if the Jacobian is \( \pm 1 \) everywhere.

**Proof.** Recall the formula of change of variable

\[
\int_E f(y) dy = \int_{\phi^{-1}(E)} f(\phi(x)) \left| \det \frac{\partial y}{\partial x} \right| dx,
\]

where \( f : \mathbb{R}^n \to \mathbb{R} \) is a continuous function. In particular, if we take \( f = 1 \), then we have

\[
\int_E dy = \int_{\phi^{-1}(E)} \left| \det \frac{\partial y}{\partial x} \right| dx,
\]

So if \( \det \frac{\partial y}{\partial x} = \pm 1 \), then we have \( m(E) = m(\phi^{-1}(E)) \) for any measurable set \( E \), where \( m \) denotes the Lebesgue measure. On the other hand, for example, if \( \det \frac{\partial y}{\partial x}(x) > 1 \) at some point \( x \), then by continuity we have that \( \det \frac{\partial y}{\partial x} > 1 \) in a neighborhood \( U \) of \( x \). Hence we get that

\[
m(\phi(U)) = \int_{\phi(U)} dy = \int_U \left| \det \frac{\partial y}{\partial x} \right| dx > \int_U dx = m(U),
\]

that is, \( \phi \) cannot be measure preserving. \( \square \)

**Proposition 16.3.** Suppose \( \phi(t, x) \) is a solution of the initial value problem

\[
\dot{x} = f(x), \quad x(0) = x,
\]

where \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a \( C^1 \) vector field. Then \( \phi(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^n \) is volume preserving if and only if \( \text{div } f = 0 \) everywhere.

**Proof.** Recall that \( \frac{\partial}{\partial x} \phi(t, x) \) satisfies the initial value problem

\[
\dot{Z} = \frac{\partial}{\partial x} f(\phi(t, x)) \cdot Z, \quad \frac{\partial}{\partial x} \phi(t, x)|_{t=t_0} = \text{id}.
\]

This is a linear equation with respect to \( Z(t) \) and of \( \frac{\partial}{\partial x} \phi(t, x) \) is in fact a fundamental matrix, and the Wronskian is \( W(t) = W(t, x) = \)
det \frac{\partial}{\partial x} \phi(t, x), which is the Jacobian of \phi(t, \cdot). Note that for any \( t_0, t \in \mathbb{R} \),

\[ W(t) = W(t_0) \exp \int_{t_0}^{t} \text{tr} \frac{\partial}{\partial x} f(\phi(s, x)) \, ds, \]

and \( \text{tr} \frac{\partial}{\partial x} f(x) = \text{div} f(x) \). So we get that for any \( t \), \( \phi(t, \cdot) \) is measure preserving if and only if \( W(t, x) = 1 \) and if and only if \( \text{div} f(x) = 0 \) everywhere.

**Proposition 16.4.** If \( X_H \) is a Hamiltonian system with Hamiltonian \( H \), and \( \phi(t, x) \) be the solution satisfying \( \phi(0, x) = x \). Then the map \( \phi(t, \cdot) : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) is volume preserving.

**Proof.** Since

\[ X_H = \left( \frac{\partial H}{\partial p_1}, \ldots, \frac{\partial H}{\partial p_n}, \frac{\partial H}{\partial q_1}, \ldots, -\frac{\partial H}{\partial q_n} \right) \]

We have

\[ \text{tr} DX_H = \sum_{i=1}^{n} \left( \frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} \right) = 0 \]

That is, \( \text{div} X_H = 0 \).

**Classical Mechanical Systems in \( \mathbb{R}^n \).**

Let \( x = (x_1, \ldots, x_n) \) denote points in \( \mathbb{R}^n \), and let \( U : \mathbb{R}^n \to \mathbb{R} \) be a \( C^1 \) function. Let \( m_i > 0, i = 1, \ldots, n \) be \( n \) positive real constants.

The system

\[ m_i \ddot{x}_i = -\frac{\partial U}{\partial x_i}, \quad i = 1, \ldots, n \]

is called a **conservative mechanical system** with **potential function** \( U \) in \( \mathbb{R}^n \). The constants represent the masses of the system, and the function \( U \) plays the role of potential energy. The system (16.3) is a formulation of Newton’s law of motion which, in words, says that \( \text{mass times acceleration equals force} \) and the force is the negative gradient of the potential energy function. Note that the potential function is a function of position alone (not velocity) and can be an arbitrary \( C^1 \) function.

Let \( c_i > 0, i = 1, \ldots, n \), denote some other constants.

The system

\[ m_i \ddot{x}_i + c_i \dot{x}_i = -\frac{\partial U}{\partial x_i}, \quad i = 1, \ldots, n, \]
is called a *dissipative mechanical system* with potential function $U$ and frictional constants $c_i$.

Given (16.3) or (16.4), we set $v = (v_1, \ldots, v_n) = (\dot{x}_1, \ldots, \dot{x}_n)$ and form the function

$$T(x, v) = \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2 + U(x).$$

This is called the *total energy function*, or simply the *energy function*, of the system. The function $K(v) = \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2$ is called the *Kinetic Energy* of the system. It is a function of velocity alone.

The equations (16.3), (16.4) are second order systems. We can form the associated first order systems

$$\dot{x}_i = v_i,$$

$$m_i \dot{v}_i = - \frac{\partial U}{\partial x_i};$$

and

$$\dot{x}_i = v_i,$$

$$m_i \dot{v}_i = - c_i v_i - \frac{\partial U}{\partial x_i}.$$

**Proposition 16.5.** There is a coordinate system on $\mathbb{R}^{2n}$ in which the conservative system (16.3) becomes a Hamiltonian system.

**Proof.** Let $q_i = x_i, p_i = m_i v_i$. Then,

$$H(q, p) = T(x, v) = \frac{1}{2} \sum_{i=1}^{n} \frac{p_i^2}{m_i} + U(q_1, \ldots, q_n)$$

and (16.3) becomes

$$\dot{q}_i = \frac{p_i}{m_i} = \frac{\partial H}{\partial p_i},$$

$$\dot{p}_i = m_i \dot{v}_i = - \frac{\partial H}{\partial q_i}.$$

**Fact 16.6.** (1) The critical points of a classical mechanical system are the points $(x, v)$ with $x$ a critical point of $U$ and $v = 0$.

(2) The total energy function $T(x, v)$ is a Lyapunov function for a conservative mechanical system and a strict Lyapunov function for a dissipative mechanical system.
(3) If \( x_0 \) is a strict relative minimum of the potential function \( U \), then \((x_0, 0)\) is a stable equilibrium of the system (16.5) and an asymptotically stable equilibrium of the system (16.6).

The fact that the energy function \( T(x, v) \) is a Lyapunov function for a mechanical system frequently helps us to get a picture of the the solutions without solving the equation.

To illustrate this phenomenon, let us consider systems with one degree of freedom. These have the form

\[
\ddot{x} + f(x) = 0,
\]

where \( f : \mathbb{R} \to \mathbb{R} \) is a real-valued function of one real variable.

Writing \( U(x) = \int_0^x f(s)ds \), we get a total energy function of the form

\[
T(x, v) = \frac{1}{2}v^2 + U(x).
\]

Let us consider some examples.

**Examples.**

1. (Harmonic oscillator) \[
T(x, v) = \frac{v^2}{2} + \frac{x^2}{2}.
\]

The orbits are circles around the origin \((0, 0)\) which is a single stable equilibrium.

2. (Pendulum) \[
T(x, v) = \frac{v^2}{2} + k(1 - \cos(x))
\]

for some constant \( k > 0 \).

The critical points are \((\pm n\pi, 0)\). The stable ones are \((2\pi n, 0)\) and the saddles are \((\pi(2n + 1), 0)\).

3. (Duffing equation) \[
T(x, v) = \frac{v^2}{2} + \frac{x^4}{4} - \frac{x^2}{2}.
\]

There are three critical points at \((0, 0), (-1, 0), (1, 0)\). The origin is a saddle and the others are centers.

When one adds friction to each of the above equations, the orbits cross the level sets of \( T \) instead of lying in them.
**Hamiltonian Systems and Variational Problems.**

We have seen that Hamiltonian systems arise naturally in Classical Mechanics. Now we will see that they also arise in general problems in the Calculus of Variations.

Consider a real-valued function $L(q, \dot{q}, t)$ of the variables $(q, \dot{q}, t) \in \mathbb{R}^{2n+1}$. Let $t_1 < t_2$ be real numbers, $a, b$ be two fixed elements in $\mathbb{R}^n$, and suppose we seek to find conditions on $C^2$ curves $\gamma : q = q(t)$ defined on the interval $[t_1, t_2]$ such that

$$q(t_1) = a, \quad q(t_2) = b$$

and

$$(16.7) \quad I(\gamma) = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t) dt$$

is stationary for nearby curves $\eta$ with the same boundary conditions $(??)$ and $\dot{q}(t)$ is the derivative $\frac{dq}{dt}(t)$.

This means that we consider one-parameter families $q(t, \alpha)$ of $C^2$ curves with $q(t, 0) = \gamma(t)$ such that

$$(16.8) \quad q(t_1, \alpha) = a, \quad q(t_2, \alpha) = b$$

for all $\alpha$ and

$$(16.9) \quad \frac{dI}{d\alpha} \bigg|_{\alpha=0} = 0$$

for

$$I(\alpha) = \int_{t_1}^{t_2} L(q(t, \alpha), \dot{q}(t, \alpha), t) dt.$$

One sometimes writes the condition (16.9) as

$$\delta \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t) dt = 0.$$

The operator $\delta$ is used to denote the fact that we are not considering an ordinary derivative, but rather, a stationary value of the integral as a family of curves changes.

Note that if $\gamma$ were a curve for which the integral (16.7) assumed a minimum for all nearby curves with the given boundary conditions, then it would be stationary in the sense of condition (16.9).

To express the derivative $\frac{dI}{d\alpha}$ more conveniently, we introduce some notation. Write $q = (q_1, \ldots, q_n), \dot{q} = (\dot{q}_1, \ldots, \dot{q}_n), L_{q_k}$ for the partial derivative of $L$ with respect to $q_k, L_{\dot{q}_k}$ for the partial derivative of $L$ with respect to $\dot{q}_k$ with $1 \leq k \leq n$. Also, we denote differentiation with respect to $t$ by “dot” and that with respect to $\alpha$ by “prime”.
Consider the condition \( \frac{dI}{dx} \bigg|_{x=0} = 0 \). We have

\[
0 = I'(\alpha) = \int_{t_1}^{t_2} L_q \cdot q' + L_{q'} \cdot q' dt
\]

where \( L_q \cdot q' \), \( L_{q'} \cdot q' \) respectively stand for

\[
\sum_{k=1}^{n} L_{q_k} q_k' \quad \text{and} \quad \sum_{k=1}^{n} L_{q_k} q_k'.
\]

By (16.8) we have \( q'(t_1, \alpha) = q'(t_1, \alpha) = 0 \). Hence \( L_q \cdot q'|_{t_1} = 0 \).

Integrating by parts we have

\[
\int_{t_1}^{t_2} L_q \cdot q' dt = L_q \cdot q'|_{t_1} - \int_{t_1}^{t_2} \frac{dL_q}{dt} \cdot q' dt = - \int_{t_1}^{t_2} \frac{dL_q}{dt} \cdot q' dt.
\]

Hence, (16.10) becomes

\[
0 = \frac{dI}{dx} \bigg|_{x=0} = \int_{t_1}^{t_2} (L_q - \frac{d}{dt} L_{q'})q' dt.
\]

Now, the “prime” derivatives \( q' \) can be made arbitrary, so the equation implies

\[
(L_q - \frac{d}{dt} L_{q'}) = 0,
\]

or, written out completely,

\[
\frac{d}{dt} L_{q_k} = L_{q_k}, \quad k = 1, \ldots, n.
\]

The equations (16.12) are called the Euler-Lagrange equations.

At a curve \((q(t), \dot{q}(t), t)\) which makes the integral (16.7) stationary, we have that \((q(t), \dot{q}(t))\) satisfies

\[
\frac{d}{dt} L_{q_k}(q(t), \dot{q}(t), t) = L_{q_k}(q(t), \dot{q}(t), t), \quad \forall k = 1, \ldots, n.
\]

Note that these are second order differential equations.

**Lemma 16.7.** The Euler-Lagrange equations have a first integral

\[
L - \dot{q} L_\dot{q}.
\]

**Proof.** This is because

\[
\frac{d}{dt} (L - \dot{q} L_\dot{q}) = L_q \ddot{q} + L_{\dot{q}} \ddot{q} - \dot{q} L_{\ddot{q}} = \dot{q} \left( L_q - \frac{dL_q}{dt} \right).
\]

\[\square\]
We now show that if in an open set $G$ in $\mathbb{R}^{2n+1}$ the matrix function

$$(16.13) \quad L_{q_k,q_l}(q,\dot{q},t)$$

is non-singular, then we can choose coordinates in which the Euler-Lagrange equations become a Hamiltonian system. This is one of the main reasons that Hamiltonian systems are important.

So, assume that we have the independent coordinates $(q,\dot{q},t)$ in an open set $G$ in $\mathbb{R}^{2n+1}$, that $L(q,\dot{q},t)$ is a $C^2$ real-valued function in $G$, and that the matrix function (16.13) is non-singular in $G$.

Consider the set of equations

$$(16.14) \quad p_k = L_{q_k}(q,\dot{q},t), \quad k = 1,\ldots,n.$$ 

Because of the assumption that (16.13) is non-singular, the Implicit Function Theorem gives us a set of $C^2$ functions $S_k(q,p,t)$ for $k = 1,\ldots,n$, such that (16.14) holds if and only if

$$(16.15) \quad \dot{q}_k = S_k(q,p,t), \quad k = 1,\ldots,n.$$ 

Let

$$H(q,p,t) = \sum_k p_k \dot{q}_k - L(q,\dot{q},t)$$

$$= \sum_k p_k S_k(q,p,t) - L(q, S(q,p,t),t).$$

Then, for $\ell = 1,\ldots,n$,

$$\frac{\partial H}{\partial p_\ell} = S_\ell(q,p,t) + \sum_k p_k \frac{\partial S_k(q,p,t)}{\partial p_\ell} - \sum_k L_{q_\ell} \frac{\partial S_k}{\partial p_\ell}$$

$$= S_\ell(q,p,t) = \dot{q}_\ell$$

since $p_k = L_{q_k}(q,\dot{q},t)$. Also,

$$\frac{\partial H}{\partial q_\ell} = -\sum_k p_k \frac{\partial S_k(q,p,t)}{\partial q_\ell} + L_{q_\ell} + \sum_k L_{q_k} \frac{\partial S_k}{\partial q_\ell}$$

$$= L_{q_\ell}$$

$$= \frac{d}{dt} \dot{q}_\ell \quad \text{(by Euler-Lagrange)}$$

$$= \frac{d}{dt} p_\ell \quad \text{(by definition of } p_\ell).$$

In the $(q,p,t)$ coordinates, we therefore have a Hamiltonian system with Hamiltonian function $H$. If $L(q,\dot{q},t) = L(q,\dot{q})$ is independent of time $t$, then so is $H$. However, in the general case, both $L$ and $H$ are time dependent. Note that if $H$ is time-dependent, then the function $H$ is not constant on solutions to the Hamiltonian system.
Let us now return to the Conservative Mechanical system with potential energy \( U \) we studied previously.

Using, position \( q \) and momentum \( p \) as coordinates, we saw that the equations of motion were a (time-independent) Hamiltonian system with Hamiltonian function

\[
H(q, p) = \frac{1}{2} \sum_k \frac{p_k^2}{m_k} + U(q_1, \ldots, q_n)
\]

and the velocity \( \dot{q} \) satisfied \( \dot{q}_k = \frac{p_k}{m_k} \).

If we assume that this Hamiltonian system comes from a variational problem as above, we are led to write

\[
H = \sum_k p_k \dot{q}_k - L(q, p),
\]

or

\[
L = \sum_k p_k \dot{q}_k - H = \sum_k p_k \dot{q}_k - \frac{1}{2} \sum_k \frac{p_k^2}{m_k} - U
\]

\[
= \sum_k p_k \frac{p_k}{m_k} - \frac{1}{2} \sum_k \frac{p_k^2}{m_k} - U = \frac{1}{2} \sum_k \frac{p_k^2}{m_k} - U = K - U,
\]

where \( K \) denotes the kinetic energy. The function \( L = K - U \) is called the **Lagrangian function** or **action function**, as opposed to the function \( T = K + U \), which was called the **Energy function**.

From the above, we are led to guess that conservative mechanical systems would satisfy the Euler-Lagrange equations for the function \( L = K - U \). This is indeed the case as can be easily verified. In this case, it can be verified that the integral

\[
\int L(q, \dot{q}) dt
\]

is actually minimized by the solution curves \( (q, \dot{q}) \), not just made stationary. This is known as Hamilton’s **Principal of Least Action**.

**Examples.**

1. (Shortest curve) Suppose that we consider the problem of finding the curve \( \gamma \) of shortest length shortest joining two points \( a, b \in \mathbb{R}^2 \).

Writing \( \gamma(t) = (x(t), y(t)) \), \( t_1 \leq t \leq t_2 \), we seek to minimize the function

\[
I(\gamma) = \int_{t_1}^{t_2} \sqrt{x'^2 + y'^2} dt
\]

over all such curves.
Let $L(x, y, \dot{x}, \dot{y}, t) = \sqrt{x^2 + y^2}$. The Euler-Lagrange equations become

$$\frac{d}{dt} L_x = L_x, \quad \frac{d}{dt} L_y = L_y.$$  

Since, $L$ is independent of $x, y, t$, we get

$$\frac{d}{dt} L_x = 0, \quad \frac{d}{dt} L_y = 0, \quad L_t = 0.$$  

These equations become

$$\frac{d}{dt} L_x = \frac{d}{dt} \frac{\dot{x}}{L} = \frac{\ddot{x} - \dot{x} L_t}{L^2} = \frac{\ddot{x}}{L} = 0,$$

and

$$\frac{d}{dt} L_y = \frac{d}{dt} \frac{\dot{y}}{L} = \frac{\ddot{y} - \dot{y} L_t}{L^2} = \frac{\ddot{y}}{L} = 0.$$  

Using that $L$ is never zero, we see that the only solutions are those $(x(t), y(t))$ for which $\ddot{x} = 0, \ddot{y} = 0$. That is, the only solutions are straight lines. With the above boundary condition, we get a unique line segment joining $a$ to $b$.

2. (Minimal surface of revolution) We consider the problem of finding the curve $\gamma$ joint two points $a, b \in \mathbb{R}^2$ that has minimal surface of revolution about the $x$-axis.

Writing $\gamma(t) = (x(t), y(t))$, $t_1 \leq t \leq t_2$, we seek to minimize the function

$$S(\gamma) = 2\pi \int_{t_1}^{t_2} y \sqrt{1 + y'^2} dx$$

over all such curves.

Replacing $t$ and $q$ by $x$ and $y$ respectively, we get

$$L(x, y, \dot{y}) = L(y, \dot{y}) = y \sqrt{1 + y'^2}.$$  

The Euler-Lagrange equation is

$$L_y = \frac{dL_y}{dx},$$

and by Lemma 16.7 it has a first integral

$$y \sqrt{1 + y'^2} - \dot{y} \cdot y \frac{\dot{y}}{\sqrt{1 + y'^2}} = c_1,$$

or

$$\frac{y}{\sqrt{1 + y'^2}} = c_1$$
for some constant $c_1$. Let $\dot{y} = \sinh \tau$, then the equation gives $y = c_1 \cosh \tau$. Hence

$$dx = \frac{dy}{y} = \frac{c_1 \sinh \tau \, d\tau}{\sinh \tau} = c_1 \, d\tau,$$

and we get $x = c_1 \tau + c_2$. So

$$y = c_1 \cosh \frac{x - c_2}{c_1}.$$

This is chain curve, and the constants $c_1$ and $c_2$ can be determined by using the fact that the curve passing through points $a, b \in \mathbb{R}^2$.

This curve is also known as a chain curve or catenary.

3. (Brachistochrone curve, or curve of fastest descent) We consider the problem of finding the curve $\gamma$ joint two points $a, b \in \mathbb{R}^2$ such that a particle will slide from $a$ to $b$ in the least amount of time.

Take coordinate system such that $a$ is at the origin, and the $y$-axis is oriented downwards, and the $x$-axis is oriented in the direction that makes the point $b$ in the first quadrant.

For a curve joint points $a$ and $b$, we take the arc length $s$ as a parameter. Since the kinetic energy of the particle is equal to the change of the potential energy, $mgy = \frac{1}{2}mv^2$, that is, $v = \sqrt{2gy}$. So we get

$$\frac{ds}{dt} = v = \sqrt{2gy},$$

or

$$dt = \frac{ds}{v} = \frac{\sqrt{1 + \dot{y}^2} \, dx}{\sqrt{2gy}}.$$ 

Hence, the time needed for the particle travelling from point $a$ to point $b$ along a curve $\gamma$ is given by

$$T(\gamma) = \frac{1}{\sqrt{2g}} \int_0^{x_b} \frac{\sqrt{1 + \dot{y}^2}}{\sqrt{y}} \, dx,$$

where $x_b$ is the $x$-coordinate of $b$. We will minimize the function over curves from $a$ to $b$.

Replacing $t$ and $q$ by $x$ and $y$ respectively, we get

$$L(x, y, \dot{y}) = \frac{\sqrt{1 + \dot{y}^2}}{\sqrt{y}}.$$ 

By Lemma 16.7 a first integral of the Euler-Lagrange equation is

$$\frac{\sqrt{1 + \dot{y}^2}}{\sqrt{y}} - \dot{y} \cdot \frac{\dot{y}}{\sqrt{y} \sqrt{1 + \dot{y}^2}} = \sqrt{c},$$
or
\[ \frac{1}{\sqrt{y\sqrt{1 + y^2}}} = \sqrt{c} \]
for some constant \( c > 0 \). We can write the equation as
\[ c = y(1 + y^2). \]

Let \( \dot{y} = \cot \tau \), then
\[ y = \frac{c}{1 + y^2} = \frac{c}{1 + \cot^2 \tau} = c \sin^2 \tau = \frac{c}{2}(1 - \cos 2\tau). \]

Also, since
\[ dx = \frac{dy}{y} = \frac{2c \sin \tau \cos \tau d\tau}{\cot \tau} = 2c \sin^2 \tau d\tau = c(1 - \cos 2\tau)d\tau, \]
we have
\[ x = c\left(\tau - \frac{1}{2} \sin 2\tau\right) + c' = \frac{c}{2}(2\tau - \sin 2\tau) + c'. \]

Since we assume that \( a \) is at the origin, \( x = 0 \) at \( \tau = 0 \). So \( c' = 0 \).

Replace \( 2\tau \) by \( \theta \) and \( c \) by \( 2A \), we have
\[ x = A(\theta - \sin \theta) \]
\[ y = A(1 - \cos \theta). \]
(16.16)

This curve is also known as a \textit{cycloid}. 