Solutions for Selected Problems in the Second Exam

1. Let \( s_1 = 4 \) and \( s_{n+1} = \sqrt{2s_n - 1} \) for \( n \geq 1 \). Show \( \lim_{n \to \infty} s_n \) exists and find the limit.

**Solution.** First we prove by induction that \( (s_n) \) is a decreasing sequence.

Since \( s_1 = 4 \) and \( s_2 = \sqrt{2 \cdot 4 - 1} = \sqrt{7} \), \( s_1 > s_2 \). Suppose \( s_k > s_{k+1} \) for some \( k \geq 1 \). Then

\[
2s_k - 1 > 2s_{k+1} - 1, \quad \text{and} \quad \sqrt{2s_k - 1} > \sqrt{2s_{k+1} - 1}.
\]

That is, \( s_{k+1} > s_{k+2} \).

By induction, \( s_n > s_{n+1} \) for all \( n \geq 1 \).

Next we prove by induction that \( (s_n) \) is bounded below by 1.

Since \( s_1 = 4 > 1 \), \( s_1 > 1 \). Suppose \( s_k > 1 \) for some \( k \geq 1 \). Then

\[
2s_k - 1 > 2 \cdot 1 - 1 = 1 \quad \text{and} \quad \sqrt{2s_k - 1} > 1.
\]

That is, \( s_{k+1} > 1 \).

By induction, \( s_n > 1 \) for all \( n \geq 1 \).

Since \( (s_n) \) is a decreasing sequence and bounded below, it converges. Denote \( s = \lim_{n \to \infty} s_n \). We skip the rest. You know how to get \( s = 1 \).

2(b). Let \( (s_n) \) be a Cauchy sequence. Use the definition to prove that the sequence \( (2s_n + 5) \) is also a Cauchy sequence.

**Proof.** Let \( \varepsilon > 0 \). Since \( (s_n) \) is a Cauchy sequence, \( \exists N > 0 \) such that for any \( m, n > N \),

\[
|s_m - s_n| < \frac{\varepsilon}{2}.
\]

Hence,

\[
|(2s_m + 5) - (2s_n + 5)| = 2|s_m - s_n| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon.
\]

By definition, \( (2s_n + 5) \) is also a Cauchy sequence.

3. Use the Bolzano-Weierstrass Theorem to show the following.

(a) (12 pts) The sequence \( (s_n) \), where \( s_n = 2 \sin^3 n + 6 \cos^3(2n) \), has a convergent subsequence.

(b) (6 pts extra) Given that \( (\cos n) \) has a subsequence converging to a nonzero real number, the sequence \( (\tan n) \) has a convergent subsequence.

**Proof.** (a) Since \( |\sin x|, |\cos x| \leq 1 \) for any \( x \in \mathbb{R} \), \( |\sin^3 n|, |\cos^3(2n)| \leq 1 \) for any \( n \in \mathbb{N} \). Hence,

\[
|s_n| = |2 \sin^3 n + 6 \cos^3(2n)| \leq 2 \sin^3 n + 6 \cos^3(2n) | \leq 2 \cdot 1 + 6 \cdot 1 = 8.
\]

So \(-8 \leq s_n \leq 8 \) for any \( n \in \mathbb{N} \), that is, \( (s_n) \) is a bounded sequence. By the Bolzano-Weierstrass Theorem, the sequence has a convergent subsequence.

(b) Let \( (\cos n_k) \) be a subsequence of \( (\cos n) \) converging to a nonzero real number \( c \). We have \( c = \lim_{k \to \infty} \cos n_k \).

Since \( (\sin n_k) \) is a bounded sequence, it has a convergent subsequence \( (\sin n_{k_j}) \). Assume \( s = \lim_{j \to \infty} \sin n_{k_j} \).

Since \( (\cos n_{k_j}) \) is a subsequence of \( (\cos n_k) \), \( c = \lim_{j \to \infty} \cos n_{k_j} \).

Since \( c \neq 0 \) and \( \tan n = \frac{\sin n}{\cos n} \), \( (\tan n_{k_j}) \) is a convergent subsequence of \( (\tan n) \) with \( \lim_{j \to \infty} \cot n_{k_j} = \frac{c}{s} \).

**Question:** How about the sequence \( (\cot n) \)?
4. Let \((s_n)\) be a sequence in \(\mathbb{R}\). Prove that for any \(\varepsilon > 0\), there exists \(N > 0\) such that \(\forall n > N\),
\[
s_n < \limsup_{n \to \infty} s_n + \varepsilon.
\]

Proof. Denote \(s^* = \limsup_{n \to \infty} s_n\). By definition, \(s^* = \lim_{N \to \infty} \sup\{s_n : n > N\}\).

Let \(\varepsilon > 0\) be given. \(\exists N_1 > 0\) such that for any \(N > N_1\),
\[
|\sup\{s_n : n > N\} - s^*| < \varepsilon.
\]

Hence, \(-\varepsilon < \sup\{s_n : n > N\} - s^* < \varepsilon\). The second inequality gives
\[
\sup\{s_n : n > N\} < s^* + \varepsilon.
\]

In particular, if we take \(N = N_1 + 1\), then by the definition of suprior, the inequality gives that for any \(n > N\),
\[
s_n \leq \sup\{s_n : n > N\} < s^* + \varepsilon.
\]

\(\Box\)

5. Prove \((s_n)\) is bounded if and only if \(\limsup_{n \to \infty} |s_n| < +\infty\).

Proof. \(\implies\) Suppose \((s_n)\) is bounded. By definition, \(\exists M > 0\) such that \(|s_n| < M\) \(\forall n \geq 0\).

By the fact that \(c < s_n - a < b\) \(\forall n \geq 0\) implies \(c \leq \limsup s_n - a < b\), we can get \(\limsup_{n \to \infty} |s_n| \leq M < +\infty\).

\(\Leftarrow\) Suppose \(\limsup_{n \to \infty} |s_n| < +\infty\). Denote \(s^* = \limsup_{n \to \infty} |s_n|\).

Let \(\varepsilon = 1\). By the above problem, \(\exists N > 0\) such that for any \(n > N\),
\[
|s_n| < s^* + 1.
\]

Take
\[
M = \max\{|s_1|, \ldots, |s_N|, s^* + 1\}.
\]

Then \(|s_n| \leq M\) \(\forall n > 0\). That is, \(-M \leq s_n \leq M\) \(\forall n > 0\). By definition, \((s_n)\) is bounded. \(\Box\)

6(b). Determine whether the series \(\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}\) converges, and find the sum if it converges. (Hint: \(\frac{1}{(2n-1)(2n+1)} = \frac{1}{2} \left[ \frac{1}{2n-1} - \frac{1}{2n+1} \right]\).)

Solution. we have
\[
a_1 = \frac{1}{2} \left[ 1 - \frac{1}{3} \right], \quad a_2 = \frac{1}{2} \left[ \frac{1}{3} - \frac{1}{5} \right], \quad \ldots, \quad a_k = \frac{1}{2} \left[ \frac{1}{2k-1} - \frac{1}{2k+1} \right], \quad \ldots
\]

So,
\[
s_1 = a_1 = \frac{1}{2} \left[ 1 - \frac{1}{3} \right], \quad s_2 = a_1 + a_2 = \frac{1}{2} \left[ 1 - \frac{1}{5} \right].
\]

Suppose \(s_k = \frac{1}{2} \left[ 1 - \frac{1}{2k+1} \right]\) for some \(k > 0\). Then
\[
s_{k+1} = s_k + a_{k+1} = \frac{1}{2} \left[ 1 - \frac{1}{2k+1} \right] + \frac{1}{2} \left[ \frac{1}{2k+3} - \frac{1}{2k+1} \right] = \frac{1}{2} \left[ 1 - \frac{1}{2k+3} \right] = \frac{1}{2} \left[ 1 - \frac{1}{2(k+1) + 1} \right].
\]

By induction, we get that \(s_n = \frac{1}{2} \left[ 1 - \frac{1}{2n+1} \right]\) \(\forall n > 0\).

Since \(\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{1}{2} \left[ 1 - \frac{1}{2n+1} \right] = \frac{1}{2}\), the series \(\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}\) converges and the sum is \(\frac{1}{2}\). \(\Box\)