1. Use definition to find the Taylor series of the function and prove convergence for all $x \in \mathbb{R}$ by showing $\lim_{n \to \infty} R_n(x) = 0$.

(a) $f(x) = \cos x$

(b) $f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$

Solution to part (b). Since $f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$, we can get $f'(x) = \frac{e^x + e^{-x}}{2} = \cosh x$, $f''(x) = \frac{e^x - e^{-x}}{2} = \sinh x$, $f'''(x) = \frac{e^x + e^{-x}}{2} = \cosh x$. In general, we have

$$f^{(2k)}(x) = \frac{e^x - e^{-x}}{2} = \sinh x, \quad \text{and} \quad f^{(2k+1)}(x) = \frac{e^x + e^{-x}}{2} = \cosh x \quad \forall k \geq 0.$$ 

Hence,

$$f^{(2k)}(0) = 0, \quad \text{and} \quad f^{(2k+1)}(0) = 1 \quad \forall k \geq 0.$$ 

So the Taylor series of $f(x)$ about $x = 0$ is

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1} = x + \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots + \frac{1}{(2k+1)!} x^{2k+1} + \ldots.$$ 

The remainder is

$$R_n(x) = \begin{cases} 
\sin y x^n = \frac{e^y - 1}{n!} \left( \frac{e^{-y}}{2} - \frac{e^{-y}}{4} + \cdots + \frac{e^{-y}}{2^n} \right) x^n \text{ if } n \text{ is even;} \\
\cos y x^n = \frac{1}{2n!} \left( \frac{e^y}{2} - \frac{e^{-y}}{4} + \cdots + \frac{e^{-y}}{2^n} \right) x^n \text{ if } n \text{ is odd.}
\end{cases}$$

where $y$ is between $x$ and 0. $y$ depends on both $x$ and $n$, and sometimes is written as $y_n$.

For any $M > 0$, if $x \in [-M, M]$, then $y \in [-M, M]$ for all $n$. Hence, both $\sinh y, \cosh y \leq \frac{e^{|y|} + e^{-|y|}}{2} \leq e^M$.

By Corollary 31.4, we get that

$$\lim_{n \to \infty} R_n(x) = 0 \quad \forall x \in (-M, M).$$

Since $M$ is arbitrary, we get that

$$\lim_{n \to \infty} R_n(x) = 0 \quad \forall x \in (-\infty, \infty).$$

Note that the convergence is uniform on $(-M, M)$, and not uniform on $(-\infty, \infty)$.

Actually you can get the limit directly without using the corollary, since $|R_n(x)| \leq \frac{e^{|y|}}{n!} x^n \to 0$ as $n \to \infty$

Since $\lim_{n \to \infty} R_n(x) = 0$ for all $x \in (-\infty, \infty)$, the Taylor’s series converges to $f(x)$ for all $x \in (-\infty, \infty)$.

For part (a), there are four cases for the remainder $R_n(x)$ since there are four possibility for $f^{(n)}(x)$: $\pm \sin x$ and $\pm \cos x$.

2. Use definition to find the Taylor series of the function and the Lagrange form of the remainder.

(a) $f(x) = (1 + x)^{1/3}$

(b) $f(x) = \frac{1}{\sqrt{1 + x}}$
Answer for part (a). \( f(x) = (1 + x)^{1/3} \), \( f'(x) = \frac{1}{3}(1 + x)^{-2/3} \), and for \( n > 1 \)

\[
 f^{(n)}(x) = \frac{(-1)^{n-1}}{3^n} 2 \cdot 5 \cdot \ldots \cdot (3n - 4)(1 + x)^{-(3n-1)/3}.
\]

You can prove it by induction. So \( f(0) = 1 \), \( f'(0) = \frac{1}{3} \), and for \( n > 1 \)

\[
 f^{(n)}(0) = \frac{(-1)^{n-1}}{3^n} 2 \cdot 5 \cdot \ldots \cdot (3n - 4).
\]

The Taylor series is

\[
 f(x) = 1 + \frac{1}{3}x + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{3^n n!} 2 \cdot 5 \cdot \ldots \cdot (3n - 4) x^n.
\]

If you want to put the term for \( n = 1 \) under the summation, you can write

\[
 f(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{3^n n!} (-1) \cdot 2 \cdot 5 \cdot \ldots \cdot (3n - 4) x^n.
\]

and the remainder for \( n \geq 2 \) is

\[
 R_n(x) = \frac{(-1)^{n-1}}{3^n n! (1 + y)^{(3n-1)/3}} 2 \cdot 5 \cdot \ldots \cdot (3n - 4)x^n.
\]

where \( y \) is between 0 and \( x \). The series converges for \( x \in [-1, 1] \). With some effort, you may be able to prove for \( x \in [0, 1] \) (not required).

3. (1) Write \( f(x) = e^x \) in the form \( f(x) = f_n(x) + R_n(x) \) for \( c = 0 \) and \( n = 3 \), where \( f_n(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!} (x - c)^k \) is the Taylor Polynomial of \( f(x) \) of order \( n \) and \( R_n(x) \) is the remainder in the Lagrange form.

(b) Use the result you get from part (a) to find \( \lim_{x \to 0} \frac{e^x - 1 - x}{x^2} \).

Answer. (a) \( f(x) = 1 + x + \frac{1}{2}x^2 + \frac{e^y}{3!}x^3 \), where \( y \) is between 0 and \( x \). (b) \( \frac{1}{2} \).