### 3.17 Limits and Continuity

**Definition. Limits**

Let $f$ be defined on $D \subset \mathbb{R}$. We say that $f(x)$ has limit $L$ as $x$ approaches $c \in \mathbb{R}$, provided that for every $\varepsilon > 0$ there is a $\delta = \delta(c, \varepsilon) > 0$ such that if $x \in D$ and

\[
0 < |x - c| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon
\]

Then we write

\[
\lim_{x \to c} f(x) = L
\]

**Remark.**

i. $c$ need not be an element in $D = \text{Dom}(f)$.

ii. Verifying (2) by way of (1) is called an $\varepsilon$-$\delta$ argument.

iii. Notice that it’s enough to show that (1) holds for some positive $\varepsilon_1$ with $\varepsilon_1 < \varepsilon_1$. For example, we sometimes assume that $1 \geq \varepsilon > 0$.

**Example 1.** Use an $\varepsilon$-$\delta$ argument to prove $\lim_{x \to 4} 3x = 12$.

**Proof.** Given $\varepsilon > 0$, we must find a $\delta > 0$ so that (1) holds. It is important to observe that $\delta$ will usually depend on the given $\varepsilon$ and the value of $c$ (4 in this problem). We claim that $\delta = \varepsilon/3$ will suffice. Now suppose that $0 < |x - 4| < \delta = \varepsilon/3$. Then

\[
|f(x) - L| = |3x - 12| = 3|x - 4| < 3 \times \frac{\varepsilon}{3} = \varepsilon
\]

□
Definition. Continuity

Let $f$ be defined on $D \subset \mathbb{R}$. We say that $f(x)$ is continuous at $c \in D$, provided that for every $\varepsilon > 0$ there is a $\delta = \delta(c, \varepsilon) > 0$ such that if $x \in D$ and

\begin{equation}
|x - c| < \delta \quad \text{then} \quad |f(x) - f(c)| < \varepsilon
\end{equation}

(3)

Equivalently, we say $f$ is continuous at $c$ provided that

\begin{equation}
\lim_{x \to c} f(x) = f(c)
\end{equation}

(4)

Compare (3) and (4) with (1) and (2). In particular, notice that it is necessary (but not sufficient) that $c \in D = \text{Dom}(f)$ in the definition of continuity.

Remark.

i. A function is simply called continuous if it is continuous at each point in its domain.

ii. If $f$ is not continuous at $c$, we say that $f$ is discontinuous at $c$ and that $c$ is a point of discontinuity of $f$.

Example 2. Let $f(x) = 3x$. Use an $\varepsilon$-$\delta$ argument to prove that $f$ is continuous at 4.

Since $f(4) = 12$ we actually proved this in Example 1. That is, we proved that

\[ \lim_{x \to 4} f(x) = 12 = f(4). \]

The following inequality is used below.

Example 3. Let $0 < \varepsilon < 1$. Prove that $\sqrt{4 + \varepsilon} - 2 < 2 - \sqrt{4 - \varepsilon}$.

Since $0 < \varepsilon < 1$ it follows that

\[ 0 < 16 - \varepsilon^2 < 16 \]
\[ \sqrt{16 - \varepsilon^2} < 4 \]
\[ 2\sqrt{16 - \varepsilon^2} < 8 \]
\[ (4 + \varepsilon) + 2\sqrt{16 - \varepsilon^2} + (4 - \varepsilon) < 16 \]
\[ (\sqrt{4 + \varepsilon} + \sqrt{4 - \varepsilon})^2 < 16 \]
\[ \sqrt{4 + \varepsilon} + \sqrt{4 - \varepsilon} < 4 \]
\[ \sqrt{4 + \varepsilon} - 2 < 2 - \sqrt{4 - \varepsilon} \]
Example 4. Prove that $g(x) = x^2$ is continuous at 2.

Proof. Given $1 > \varepsilon > 0$. We let $\delta = \sqrt{4 + \varepsilon} - 2$ (which is positive since $\varepsilon > 0$). Now to verify (3), we consider two cases. First suppose that $2 < x < 2 + \delta$. Then

\begin{equation}
0 < x - 2 < \delta = \sqrt{4 + \varepsilon} - 2
\end{equation}

so that

\begin{align*}
2 &< x < \sqrt{4 + \varepsilon} \\
4 &< x^2 < 4 + \varepsilon \\
0 &< x^2 - 4 < \varepsilon
\end{align*}

Now suppose that $2 - \delta < x < 2$. Rearranging yields

\begin{equation}
0 < 2 - x < \delta
\end{equation}

Now by Example 3, $\delta < 2 - \sqrt{4 - \varepsilon}$. Thus

\begin{align*}
0 < 2 - x < 2 - \sqrt{4 - \varepsilon} &\iff -2 < -x < -\sqrt{4 - \varepsilon} \\
\end{align*}

Rearranging yields

\begin{align*}
\sqrt{4 - \varepsilon} &< x < 2 \\
4 - \varepsilon &< x^2 < 4 \\
- \varepsilon &< x^2 - 4 < 0
\end{align*}

Now (5) and (6) are equivalent to $0 < |x - 2| < \delta$. We have shown that

\begin{equation*}
0 < |x - 2| < \delta \implies |x^2 - 4| < \varepsilon.
\end{equation*}

Since $g(2) = 4$ we have proven that if $|x - 2| < \delta$, then

\begin{equation*}
|g(x) - g(2)| = |x^2 - 4| < \varepsilon
\end{equation*}

as desired. \qed
Here is an easier argument. Notice that if \( x \) is “near” 2, then \( x + 2 \) should be bounded (above and below). More precisely, if \( 0 < \delta \leq 1 \) say, then \( |x - 2| < \delta \) implies

\[
-\delta < x - 2 < \delta \\
\implies 4 - \delta < x + 2 < 4 + \delta < 5 \\
\implies |x + 2| < 5
\]

Now let’s revisit the previous example. Once again show that \( g(x) = x^2 \) is continuous at 2.

**Proof.** Let \( \varepsilon > 0 \). Now choose \( \delta = \min\{1, \varepsilon/5\} \). Then if \( x \) satisfies \( |x - 2| < \delta \) we have

\[
|x^2 - 4| = |x + 2||x - 2| \\
\leq 5|x - 2| \quad \text{(since \( \delta \leq 1 \))} \\
< 5 \left( \frac{\varepsilon}{5} \right) \quad \text{(since \( \delta \leq \varepsilon/5 \))} \\
= \varepsilon
\]

\( \square \)

**Remark.** Notice that the number 5 was unimportant. The important point is that \( |x + 2| \) is bounded by some \( M > 0 \).
**Theorem 1.** Let $f$ be a continuous function and $k$ be any real number. Then $k \cdot f$ and $|f|$ are also continuous functions.

The following proposition will be used frequently when dealing with limits of continuous functions.

**Proposition 2.** Suppose that $f$ is continuous at $c$ and that $f(c) > L$ for some $L \in \mathbb{R}$. Then there is a $\delta > 0$ such that $f(x) > L$ for all $x \in (c - \delta, c + \delta) \cap \text{Dom}(f)$.

**Proof.** By considering the continuous function $f(x) - L$, it is enough to prove the special case when $L = 0$. Now let $\varepsilon = f(c)/2 > 0$. Since $f$ is continuous at $c$, there is a $\delta > 0$ such that $|x - c| < \delta$ and $x \in \text{Dom}(f)$ implies

$$-\varepsilon < f(x) - f(c) < \varepsilon$$

Focusing on the left inequality we see that

$$-\frac{f(c)}{2} < f(x) - f(c)$$

In other words

$$f(x) > \frac{f(c)}{2}$$

\[ \square \]

The next result will help us create new continuous functions from old ones.

**Theorem 3.** Suppose that $f$ and $g$ are continuous at $c \in \text{Dom}(f) \cap \text{Dom}(g)$. Then so are $f \pm g$, $fg$, and $f/g$ (provided $g(c) \neq 0$).

**Proof.** You can find proofs of the first two results in the text. The proof that $f/g$ is continuous will be broken up into 3 parts. The motivation for the (sometimes) mysterious choices for $\varepsilon$ below was presented during class.

**Part 1.** Apply Theorem 2 to the continuous function $|g(x)|$ to conclude there is $\delta_0 > 0$ such that $|g(x)| > |g(c)|/2 > 0$ whenever $|x - c| < \delta_0$. It follows that

\begin{equation}
|x - c| < \delta_0 \implies \frac{1}{|g(x)|} < \frac{2}{|g(c)|}
\end{equation}
Part 2. Now let $\varepsilon > 0$. By the continuity of $f$, there is a $\delta_1 > 0$ such that

$$|f(x) - f(c)| < \frac{|g(c)|\varepsilon}{4}$$

whenever $|x - c| < \delta_1$. Also, by the continuity of $g$, there is a $\delta_2 > 0$ such that

$$|g(x) - g(c)| < \frac{|g(c)|^2 \varepsilon}{4(|f(c)| + 1)}$$

whenever $|x - c| < \delta_2$.

Part 3. Now let $\delta = \min\{\delta_0, \delta_1, \delta_2\}$. Then $|x - c| < \delta$ implies

$$\left| \frac{f(x)}{g(x)} - \frac{f(c)}{g(c)} \right| = \left| \frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)} \right|$$

$$= \frac{1}{|g(x)g(c)|} |f(x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(x)|$$

$$< \frac{2}{|g(c)|^2} |f(x)g(c) - f(c)g(c)| + |f(c)g(c) - f(c)g(x)|$$

$$= \frac{2|g(c)|}{|g(c)|^2} |f(x) - f(c)| + \frac{2|f(c)|}{|g(c)|^2} |g(c) - g(x)|$$

$$< \frac{2}{|g(c)|} \left( \frac{|g(c)|\varepsilon}{4} + \frac{2|f(c)|}{|g(c)|^2} \frac{|g(c)|^2 \varepsilon}{4(|f(c)| + 1)} \right)$$

$$= \frac{\varepsilon}{2} + \frac{|f(c)|\varepsilon}{2(|f(c)| + 1)}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

Line (8) follows from Part 1 since $\delta < \delta_0$. Line (9) follows from Part 2 since $\delta < \delta_1$ and $\delta < \delta_2$. Finally, line (10) follows since

$$\frac{|f(c)|}{2(|f(c)| + 1)} < 1$$