Convergent Sequences

A sequence \( \{a_n\} \) is **bounded** if there is a real number \( M \) such that \( |a_n| \leq M \) for all \( n \in \mathbb{N} \).

**Theorem** Convergent sequences are bounded.

**Proof:** Let \( \{a_n\} \) be a convergent sequence with limit \( s \) and let \( \varepsilon = 623 \). Then there exists a natural number \( N \) such that

\[
    n > N \quad \text{implies} \quad |s_n - s| < 623
\]

(1)

Thus

\[
    |s_n| = |s_n - s + s| \leq |s_n - s| + |s| \leq 623 + |s|
\]

for all \( n > N \).

Now let \( M \) be the maximum of the finite set

\[
    \{623 + |s|, |a_1|, |a_2|, \ldots, |a_N|\}.
\]

Then \( |a_n| \leq M \) for all \( n \in \mathbb{N} \), as desired.

**Note:** The last proof makes use of a very important idea:

All but a finite number of terms in a convergent sequence are arbitrarily close to the limit.
We will exploit this idea again below.

**Proposition**

Suppose that \( \lim_{n \to \infty} b_n = b \neq 0 \). Then there is a natural number \( N \) such that for all \( n > N \), \( b_n \neq 0 \).

**Proof:** Without loss of generality we may assume \( b > 0 \). Now let \( \varepsilon = \frac{b}{2} > 0 \). So there is an \( N \in \mathbb{N} \) such that for all \( n > N \), \( |b_n - b| < \frac{b}{2} \). Rearranging we see that this implies \( b_n > \frac{b}{2} > 0 \), as desired.

*Note:* The proof is similar if \( b < 0 \). This proposition is used in the next theorem.

### The Limit Laws

**Theorem**

Suppose that \( \lim_{n \to \infty} a_n = a \) and \( \lim_{n \to \infty} b_n = b \) and let \( k \) be a real number. Then

(a) \( \lim_{n \to \infty} ka_n = ka \)

(b) \( \lim_{n \to \infty} (a_n + b_n) = a + b \)

(c) \( \lim_{n \to \infty} (a_n b_n) = ab \)

(d) \( \lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) = \frac{a}{b}, \quad b \neq 0, \ b_n \neq 0 \) for all \( n \in \mathbb{N} \)

Before proving (d), let’s look at an example.

*Example:* Show that \( \lim_{n \to \infty} a_n = 0 \) if and only if \( \lim_{n \to \infty} a_n^2 = 0 \).

**Proof:** If \( \lim_{n \to \infty} a_n = L \neq 0 \), then by (c), \( \lim_{n \to \infty} a_n^2 = L^2 \neq 0 \). This establishes the right to left implication. We leave the forward implication as an easy exercise.

### The Limit Laws (cont)

To prove property (d), we first note that by the previous proposition, there exists \( N_1 \in \mathbb{N} \), such that for all \( n > N_1 \), \( |b_n| > \frac{|b|}{2} \). Now let \( \varepsilon > 0 \). There exists \( N_2, N_3 \in \mathbb{N} \) such that...
\[ |a_n - a| < \frac{\varepsilon |b|}{4}, \text{ provided } n > N_2 \]
\[ |b_n - b| < \frac{\varepsilon b^2}{4|a|}, \text{ provided } n > N_3 \]

Now let \( n > N = \max\{N_1, N_2, N_3\} \), then

\[
\left| \frac{a_n}{b_n} - \frac{a}{b} \right| = \frac{|a_n b - ab_n|}{|b_n b|} = \frac{|a_n b - ab + ab - ab_n|}{|b_n b|}
\leq \frac{1}{|b_n b|} (|b||a_n - a| + |a||b - b_n|)
< \frac{2}{b^2} (|b||a_n - a| + |a||b - b_n|)
< \frac{2|b|}{b^2} \left( \frac{\varepsilon |b|}{4} \right) + \frac{2|a|}{b^2} \left( \frac{\varepsilon b^2}{4|a|} \right) = \varepsilon
\]

See the text for the proofs of the other 3 properties.

*Note:* There is a mistake (call it an omission) in the above proof. Can you find it?

**Basic Examples**

**Theorem**

(a) \( \lim_{n \to \infty} \frac{1}{n^p} = 0 \) for \( p > 0 \).

(b) \( \lim_{n \to \infty} a^n = 0 \) if \( |a| < 1 \).

(c) \( \lim_{n \to \infty} n^{1/n} = 1 \).

(d) \( \lim_{n \to \infty} a^{1/n} = 1 \) for \( a > 0 \).

We prove (c) below. See the text for the remaining proofs.

**Basic Examples (cont)**

To prove (c), we let \( a_n = n^{1/n} - 1 \) and notice that \( a_n > 0 \) for \( n > 1 \). Rearranging we obtain
\[ n^{1/n} = 1 + a_n \]
\[ \Rightarrow n = (1 + a_n)^n \]
\[ = 1 + na_n + \frac{n(n-1)}{2} a_n^2 + \text{positive terms} \]
\[ > 1 + \frac{n(n-1)}{2} a_n^2 \]

It follows that \( 0 < a_n^2 < 2/n \) so that \( a_n^2 \to 0 \) as \( n \to \infty \) by the Squeeze Law (see exercise 8.5). Notice that by the above example, that \( a_n \to 0 \). Thus

\[ \lim_{n \to \infty} n^{1/n} = \lim_{n \to \infty} (n^{1/n} - 1 + 1) \]
\[ = \lim_{n \to \infty} a_n + \lim_{n \to \infty} 1 \]
\[ = 0 + 1 \]

**Infinite Limits**

**Definition** We write \( \lim_{n \to \infty} a_n = \infty \) provided that for each \( M > 0 \) there exists an \( N \) such that \( n > N \) implies \( a_n > M \).

Roughly speaking, the above definition suggests that the terms in the sequence eventually exceed any upper bound. Such limits are said to diverge to infinity. **Note:** There is a similar definition for diverging to negative infinity. See the text.

Here is a useful characterization.

**Theorem** Let \( a_n \) be a sequence of positive numbers. Then \( \lim_{n \to \infty} a_n = \infty \) if and only if \( \lim_{n \to \infty} 1/a_n = 0 \).

See the text for a proof.

**Example** \( \lim_{n \to \infty} a^n = \infty \) for \( a > 1 \).

Observe that if \( a > 1 \) then \( 1/a < 1 \) and we could prove this by appealing to the last theorem and Part b from the example above. However, with Bernoulli’s inequality, the direct proof is almost trivial.

Write \( a = 1 + c \) where \( c > 0 \). Then by Bernoulli’s Inequality we have
\[ a^n = (1 + c)^n > 1 + nc \]

Now let \( M > 1 \). By the Archimedian Property, there is a natural number \( N \) such that \( Nc > M - 1 \). It follows that for all \( n > N \)

\[
\begin{align*}
    a^n &= (1 + c)^n > 1 + nc \\
    &> 1 + Nc \\
    &> M
\end{align*}
\]
as desired.

Notice that together with the useful characterization above, this last result now establishes Part b from the basic examples theorem.