Let $f$ be a continuous function on $I = [a, b]$. Show that the function $f^*$ defined by

$$f^*(x) = \sup\{f(y) : a \leq y \leq x\},$$

for all $x \in [a, b]$, is an increasing continuous function on $[a, b]$. See Figure 1.

![Figure 1: Graph of $f$ versus $f^*$](image)

First observe that if $A \leq B \leq C \leq D$, then, for example,

$$D - A \geq C - B \geq 0, \text{ etc.}$$

We also note that since $f$ is continuous and $[a, x]$ closed and bounded for any $x \in [a, b]$, that the supremum is actually a maximum (by the Max-Min Theorem). In other words, for each $x \in [a, b]$, $\exists x_0 \in [a, x]$ such that

(2) \hspace{1cm} f^*(x) = f(x_0)

Finally, note that $f^*(x) \geq f(x)$ for all $x \in [a, b]$.

Let $a \leq s < t \leq b$. Then

$$\{f(y) : a \leq y \leq s\} \subset \{f(y) : a \leq y \leq t\}$$

So by exercise 1.4.7 from the text

$$f^*(s) = \sup\{f(y) : a \leq y \leq s\} \leq \sup\{f(y) : a \leq y \leq t\} = f^*(t)$$

In other words, $f^*$ is increasing.

**Method 1 - Show $f^*$ is (Pointwise) Continuous:**

Now let $\varepsilon > 0$ and fix $c \in [a, b]$. Then by the continuity of $f$ (at $c$), there is a $\delta = \delta(c) > 0$ such that

$$|x - c < \delta \implies |f(x) - f(c)| < \varepsilon/2$$

**Case 1:** $c < x < c + \delta$. In this case, $f^*(x) \geq f^*(c)$ since $f^*$ is increasing. If we have equality, there is nothing to show. Otherwise, we suppose that $f^*(x) > f^*(c)$. So by (2), $f^*(x) = f(x_0)$ for some $a \leq x_0 \leq x$. However, we must have $x \geq x_0 > c$ else $f^*(x) = f^*(c)$. Thus, $|x_0 - c| < \delta$. Now

$$f(x_0) = f^*(x) > f^*(c) \geq f(c)$$

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So by (1),
\[ |f^*(x) - f^*(c)| \leq |f(x_0) - f(c)| < \varepsilon/2 < \varepsilon \]

**Case 2:** \( c - \delta < x < c \). In this case, \( f^*(x) \leq f^*(c) \) since \( f^* \) is increasing. If we have equality, there is nothing to show. Otherwise, we suppose that \( f^*(x) < f^*(c) \). So by (2), \( f^*(c) = f(c_0) \) for some \( a \leq c_0 \leq c \). However, we must have \( x < c_0 \leq c \) else \( f^*(x) = f^*(c) \). In other words, we may assume that \( |x - c_0| < \delta \). Now
\[ f(x) \leq f^*(x) < f^*(c) = f(c_0) \]

So by (1),
\[ |f^*(x) - f^*(c)| \leq |f(x) - f(c_0)| \]

And now we are stuck. Can you explain why? Think about this before proceeding.

The problem is that our earlier choice of \( \delta > 0 \) specifically depended on \( c \). We don’t (immediately) know anything about the \( \delta = \delta(c_0) > 0 \) required to control the last expression above. Although we can circumvent this by appealing to the uniform continuity of \( f \) (since \( I \) is closed and bounded), there is an easier way. Notice that since \( c - \delta < x < c_0 \leq c \), we also have \( |c_0 - c| < \delta \). Now continuing with (3),
\[ |f^*(x) - f^*(c)| \leq |f(x) - f(c_0)| \]
\[ \leq |f(x) - f(c)| + |f(c) - f(c_0)| \]
\[ < \varepsilon/2 + \varepsilon/2 \]

It follows that \( f^* \) is continuous at \( c \).
Method 2: Prove $f^*$ is Uniformly Continuous:

By Theorem 8 (in class), $f$ is uniformly continuous. So let $\varepsilon > 0$. There exists a $\delta > 0$ so that $|y - x| < \delta$ implies $|f(y) - f(x)| < \varepsilon$.

Now let $y > x$ with $|y - x| < \delta$. If $f^*(y) = f^*(x)$ there is nothing to prove. Suppose then that $f^*(y) > f^*(x)$. So there exists $y_0$ with $x < y_0 \leq y$ such that $f^*(y) = f(y_0)$. (Why must $y_0 > x$?)

Also, there exists $x_0 \leq x$ such that $f^*(x) = f(x_0)$. We have

$$x_0 \leq x < y_0 \leq y$$

If $x = x_0$ we are done since we now have $|x_0 - y_0| \leq |x - y| < \delta$. So by the uniform continuity of $f$

$$|f^*(x) - f^*(y)| = |f(x_0) - f(y_0)| < \varepsilon$$

On the other hand, suppose that $x_0 < x < y_0 \leq y$. Then $f(x) < f(x_0) < f(y_0)$. By the IVP there exists $c \in (x, y_0)$ such that $f(c) = f(x_0)$ (see Figure 2). It follows that

$$|f^*(x) - f^*(y)| = |f(c) - f(y_0)| < \varepsilon$$

since $|c - y_0| < |x - y| < \delta$. Now since uniform continuity implies continuity, we are done.

Remark. Method 2 is easier to follow and it illustrates a very nice use of the IVP.