Let $a > 0$ and $b > 0$. Use an $\varepsilon$-$N$ argument to prove the limit below. Note: This is one of the more annoying forms.

$$
\lim_{n \to \infty} \frac{n}{an^2 - b} = 0
$$

**Proof.** We want to show that we can choose $n$ large enough so that $\left| \frac{n}{an^2 - b} \right| < \varepsilon$. If $n > \sqrt{b/a}$ then the denominator is positive and we can drop the absolute values below. So

$$
\left| \frac{n}{an^2 - b} \right| = \frac{n}{an^2 - b} < \varepsilon
$$

$$
\Rightarrow \quad \frac{1}{\varepsilon} < \frac{an^2 - b}{n} = an - b \frac{1}{n}
$$

Rearranging we see that we must choose $N$ large enough so that $n > N$ implies

$$
n > \frac{1}{a} \left( \frac{1}{\varepsilon} + b \frac{1}{n} \right)
$$

But what about the $n$ on the right-hand side? Observe that $b \geq b/n$ for all $n \in \mathbb{N}$. We are now in position to complete the proof. Let $\varepsilon > 0$ and let $N = \max \left\{ \sqrt{b/a}, \frac{1}{a} \left( \frac{1}{\varepsilon} + b \right) \right\}$. Then $n > N$ implies

$$
an > \frac{1}{\varepsilon} + b \Rightarrow \frac{1}{\varepsilon} + \frac{b}{n}
$$

Rearranging we see that

$$
\frac{1}{\varepsilon} < an - b \frac{1}{n} = \frac{an^2 - b}{n}
$$

Since everything is positive we can recipricate to obtain

$$
\varepsilon > \frac{n}{an^2 - b} = \left| \frac{n}{an^2 - b} - 0 \right|
$$

as desired. $\Box$

**Remark.** This proof is made incredibly annoying because of the “subtraction” in the denominator. Indeed, under the same assumptions above, observe that

$$
\frac{n}{an^2 + b} < \frac{n}{an^2} = \frac{1}{an} < \varepsilon
$$

So now for a given $\varepsilon > 0$ we can choose $N = \frac{1}{a\varepsilon}$ so that $n > N$ implies $\left| \frac{n}{an^2 + b} - 0 \right| < \varepsilon$. In other words,

$$
\lim_{n \to \infty} \frac{n}{an^2 + b} = 0$$