Series Tests for Convergence - Summary

Recall

Definition. Given the infinite series

\[ \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots \]

we define the following. The number \( a_n \) is called the \textit{nth term} of the series. It is also called the \textit{summand}. The \textit{nth partial sum} of the series is denoted by \( s_n \) and is defined by

\[ s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k \]

Notice that the partial sums generate a new sequence, the so-called \textit{sequence of partial sums}, \( \{s_n\} \). Now if this new sequence converges to a limit, say \( L \in \mathbb{R} \), we say that the series (1) converges and that its \textit{sum} is \( L \). Specifically,

\[ s_n \to L \text{ as } n \to \infty \implies \sum_{n=1}^{\infty} a_n = L \]

In other words,

\[ \sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \sum_{k=1}^{n} a_k = \lim_{n \to \infty} s_n \]

whenever the limit exists. Otherwise, the series \textit{diverges}.

We have the following general test for convergence.

**Theorem 1. Cauchy Criterion for Series.** The series \( \sum_{n=1}^{\infty} a_n \) converges if and only if, for every \( \varepsilon > 0 \) there exists an \( N \in \mathbb{N} \) such that for all \( n > m \geq N \) we have

\[ |a_{m+1} + a_{m+2} + \cdots + a_n| = \left| \sum_{j=m+1}^{n} a_j \right| < \varepsilon \]

**Proof.** Notice that

\[ s_n - s_m = \sum_{j=1}^{n} a_j - \sum_{j=1}^{m} a_j = a_{m+1} + a_{m+2} + \cdots + a_n \]

Now apply the Cauchy Criterion for sequences to \( \{s_n\} \).

We summarize the various convergence \textit{tests} for infinite series. Suppose that \( a_n \geq 0 \) for all \( n \geq N \), \( (N \in \mathbb{Z}) \). To test the series \( \sum a_n \) for convergence (or divergence) we have the following.

1. \textit{n-Term Test (for Divergence).}
   
   If \( a_n \not\to 0 \) then \( \sum a_n \) diverges.

   **Remark.** This test is valid for any series, not just series with nonnegative terms.

2. \textit{Cauchy Condensation Test.} If \( \{a_n\} \) is a nonincreasing sequence that converges to \( 0 \). Then

   \[ \sum_{n} a_n < \infty \iff \sum_{n} 2^n a_{2^n} < \infty \]
3. Comparison Test.

(a) $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \leq c_n$ for all $n \geq N$ for some positive integer $N$.

(b) $\sum a_n$ diverges if there is a divergent series $\sum d_n$ with $a_n \geq d_n \geq 0$ for all $n \geq N$ for some positive integer $N$.

4. Limit Comparison Test. Let $a_n > 0$ and $b_n > 0$ for all $n \geq N$.

(a) Suppose that $\frac{a_n}{b_n} \to \delta \in [0, \infty)$. If $\sum b_n$ converges then so does $\sum a_n$.

(b) Suppose that $\frac{a_n}{b_n} \to \delta \in (0, \infty]$. If $\sum b_n$ diverges then so does $\sum a_n$.

5. Ratio Test. Let $\sum a_n$ be a series of positive terms and suppose that

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \rho.$$ 

Then

(a) the series converges if $\rho < 1$,

(b) the series diverges if $\rho > 1$ or $\rho$ is infinite,

(c) the test is inconclusive if $\rho = 1$.

6. Root Test. Suppose that $a_n \geq 0$ for $n \geq N$ and

$$\lim_{n \to \infty} \sqrt[n]{a_n} = \rho$$

Then

(a) the series converges if $\rho < 1$,

(b) the series diverges if $\rho > 1$ or $\rho$ is infinite,

(c) the test is inconclusive if $\rho = 1$.

7. Alternating Series Test (Leibnitz's Theorem). Let $N$ be a positive integer. The alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

converges provided that the following three conditions are satisfied.

(a) $a_n > 0$ for all $n \geq N$.

(b) $a_n \geq a_{n+1}$ for all $n \geq N$.

(c) $a_n \to 0$.

Example 1. Does the series below converge or diverge. Give reasons for your answer.

$$\sum_{n=2}^{\infty} \frac{1}{1 + (\ln n)^3}$$

We claim that the series diverges by the Cauchy Condensation Test. Let

$$a_n = \frac{1}{1 + (\ln n)^3}$$

Notice that $a_n \to 0$ and, since the denominator is increasing, we clearly have $a_n \geq a_{n+1}$ so that the CCT applies. So the series $\sum a_n$ and the series $\sum 2^n a_{2^n}$ converge or diverge together. Now

$$\sum_{n=2}^{\infty} 2^n a_{2^n} = \sum_{n=2}^{\infty} \frac{2^n}{1 + (\ln 2^n)^3} = \sum_{n=2}^{\infty} \frac{2^n}{1 + (\ln 2)^3 n^3}$$

but

$$\lim_{n \to \infty} \frac{2^n}{1 + (\ln 2)^3 n^3} = \infty$$

So the series $\sum 2^n a_{2^n}$ diverges by the nth-term test. The result follows.
Example 2. Do the following series converge or diverge. Justify your claim.

a. \[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 9}} \]

b. \[ \sum_{n=1}^{\infty} \frac{n + 1}{n^2} \]

c. \[ \sum_{n=1}^{\infty} \frac{1}{3^{n-1} + 2} \]

d. \[ \sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3} \]

e. \[ \sum_{n=1}^{\infty} \frac{2^n}{(2n)!} \]

f. \[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n!)^2}{(2n)!} \]

Example 3. Do the following series converge or diverge. Justify your claim.

a. \[ \sum_{n=1}^{\infty} n e^{-n^2} \]

b. \[ \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 2} \]

c. \[ \sum_{n=1}^{\infty} \left( \frac{n}{n + 1} \right)^n \]

d. \[ \sum_{n=1}^{\infty} \frac{\cos(1/n)}{n^2} \]

e. \[ \sum_{n=1}^{\infty} \frac{3^n n!}{(2n)!} \]

f. \[ \sum_{n=1}^{\infty} (1 - \cos(1/n)) \]