16.8 Stokes' Theorem

Theorem 1. Stokes' Theorem

The circulation of $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ around the boundary C of an oriented surface S in the direction counterclockwise to the surface's unit normal vector $\mathbf{n}$ is equal to the integral

(1) $$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS$$

The theorem holds under suitable conditions. The usual conditions are that all functions and all derivatives are continuous.

Remark. Notice that the right-hand side of (1) is just the surface integral of the real-valued function

$$g = \nabla \times \mathbf{F} \cdot \mathbf{n}.$$

Now let $\mathbf{G} = \nabla \times \mathbf{F}$. Then the right-hand side of (1) can also be viewed as the *flux* of the curl since

$$\begin{aligned}\text{flux} &= \iint_S \mathbf{G} \cdot \mathbf{n}\, dS \\ &= \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS\end{aligned}$$

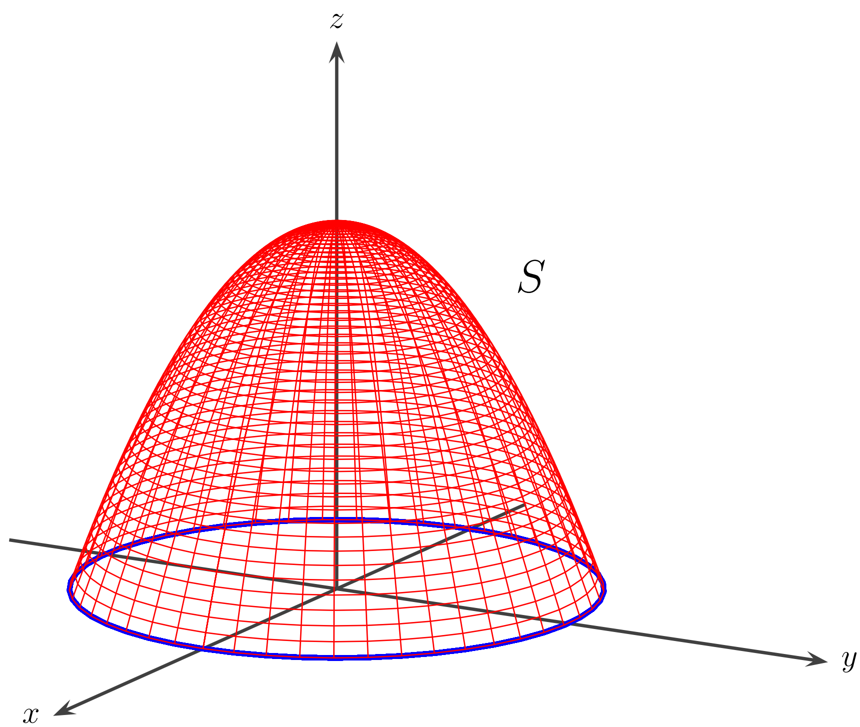

Figure 1: A parabolic cap

Example 1. Let $\mathbf{F} = 2y\,\mathbf{i} - 3x\,\mathbf{j} - z^2\,\mathbf{k}$. Let S (see Fig. 1) be the level surface of $g(x, y, z) = x^2 + y^2 + z = 9, \;\; z \geq 0$. Evaluate the surface integral below using several different methods. Orient the surface so that the vector normal has a positive $\mathbf{k}$ component.

(2)
$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS$$

(a) **Direct Computation**

Observe that the vector equation for S is given by

$$\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + (9 - x^2 - y^2)\,\mathbf{k},\ (x,y) \in R$$

where $R = \{(x,y)\,|\,x^2 + y^2 \le 9\}$. Now

$$\mathbf{r}_x = \mathbf{i} - 2x\,\mathbf{k} \quad \text{and} \quad \mathbf{r}_y = \mathbf{j} - 2y\,\mathbf{k}$$

so that

$$\mathbf{r}_x \times \mathbf{r}_y = 2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k}$$

Also, it is easy to confirm that

$$\nabla \times \mathbf{F} = -5\,\mathbf{k}$$

so that

$$\begin{aligned}\nabla \times \mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) &= -5\,\mathbf{k} \cdot (2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k}) \\ &= -5\end{aligned}$$

It follows that

$$\begin{aligned}\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\,dS &= \iint_R \nabla \times \mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y)\ dA \\ &= \iint_R -5\,dA \\ &= -5 \iint_R dA \\ &= -5 \times \text{area of } R \\ &= -5 \times 9\pi\end{aligned}$$

(b) **Using Stokes' Theorem**

Notice that the boundary of S is closed curve C which lives in the xy-plane. We first compute the (counterclockwise) **circulation** around the closed curve C which has the vector equation

$$C: \quad \mathbf{r}(t) = 3\cos t\ \mathbf{i} + 3\sin t\ \mathbf{j}, \quad 0 \le t \le 2\pi$$

Thus

$$\begin{aligned} d\mathbf{r}(t) &= -3\sin t\, dt\ \mathbf{i} + 3\cos t\, dt\ \mathbf{j} \\ \mathbf{F} &= 2y\,\mathbf{i} - 3x\,\mathbf{j} \\ \mathbf{F}(\mathbf{r}(t)) &= 6\sin t\ \mathbf{i} - 9\cos t\ \mathbf{j} \end{aligned}$$

so that

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} &= (-18\sin^2 t - 27\cos^2 t)\, dt \\ &= (-18 - 9\cos^2 t)\, dt \end{aligned}$$

Now by Stokes' Theorem (1)

$$\begin{aligned} \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS &= \oint_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^{2\pi} \left(-18 - 9\cos^2 t\right)\ dt \\ &= -36\pi - 9\pi \\ &= -45\pi \end{aligned}$$

in agreement with part (a).

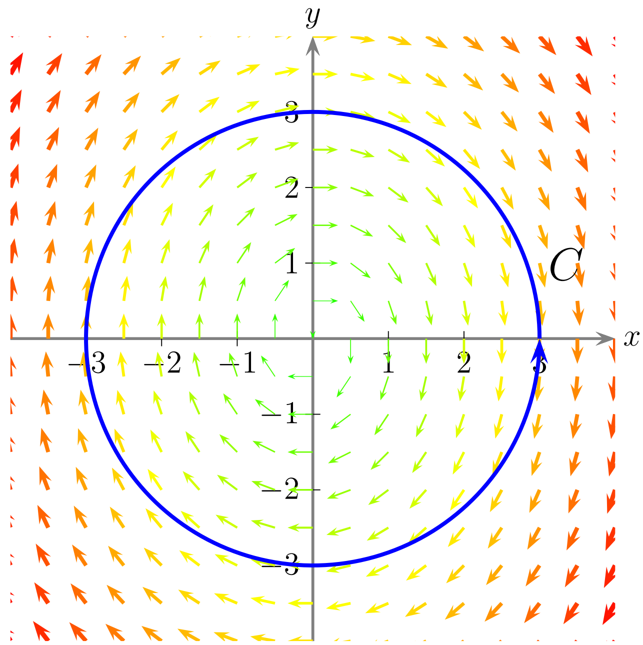

Figure 2: $\mathbf{F}\Big|_{z=0} = 2y\,\mathbf{i} - 3x\,\mathbf{j}$

(c) **Exploiting Green's Theorem**

As we observed above, the boundary of S happens to lie in the xy-plane (see Fig. 2). Now let R be as indicated in part a. Then by Stokes' Theorem and (the tangential form of) Green's Theorem, we have

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dS$$

Now by part a, $\nabla \times \mathbf{F} \cdot \mathbf{k} = -5$. Hence

$$\begin{aligned}\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA \\ &= -5 \iint_R dA \\ &= -5 \times \text{area of } R = -45\pi\end{aligned}$$

as we saw above.

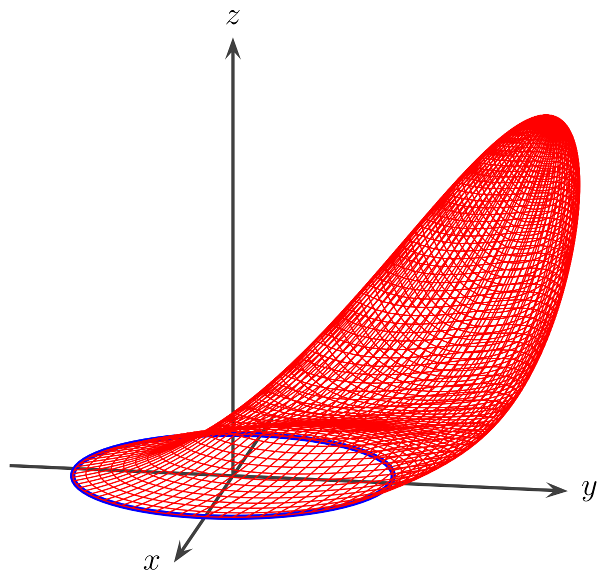

Figure 3: Continuously deformed “parabolic cap” from Example 1

Example 2. Let $\mathbf{F} = 2y\,\mathbf{i} - 3x\,\mathbf{j} - z^2\,\mathbf{k}$ be the vector field from the previous example and let S' be the surface shown in Figure 3. Notice that S' has the same boundary $C:\ \mathbf{r}(t) = 3\cos t\ \mathbf{i} + 3\sin t\ \mathbf{j},\ 0 \le t \le 2\pi$.

Then by Stokes' Theorem

$$\iint_{S'} \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

$$= -45\pi$$

The following identity has wide applications.

An Important Identity

$$\text{curl grad } f = \mathbf{0}$$

or

$$\nabla \times \nabla f = \mathbf{0}$$

Notice that the RHS is a vector. The identity is easy to prove if $f(x, y, z)$ has continuous second partials (see the text).

Example 3. Let C be the boundary of any smooth orientable surface S in space. Show that the circulation of $\mathbf{F} = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$ around C is zero.

Although we can compute $\nabla \times \mathbf{F}$ directly, we'll try another approach. Let $f(x, y, z) = x^2 + y^2 + z^2$ then $\nabla f = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k} = \mathbf{F}$ and

$$\nabla \times \mathbf{F} = \nabla \times \nabla f = \mathbf{0}$$

Now by Stokes' Theorem the circulation of $\mathbf{F}$ around C is

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS \\ &= \iint_S \mathbf{0} \cdot \mathbf{n}\, dS \\ &= 0 \end{aligned}$$

Example 4. Recognizing Integrals

Suppose that C is the smooth boundary of the region R or the orientable surface S. Identify each of the following integrals as either a flux or flow integral (or neither). Also, give any other useful information.

(a) $\oint_C 4xy\,dx - 3x\,dy$

This integral can be interpreted in two (equivalent) ways.

As a flow integral (circulation)

Let

$$\mathbf{F} = 4xy\,\mathbf{i} - 3x\,\mathbf{j}$$

then

$$\begin{aligned}\oint_R \mathbf{F}\cdot d\mathbf{r} &= \oint_C 4xy\,dx - 3x\,dy \\ &= \iint_R \left(\frac{\partial(-3x)}{\partial x} - \frac{\partial(4xy)}{\partial y}\right)\,dA \\ &= \iint_R (-3-4x)\,dA\end{aligned}$$

by Green's Theorem.

As a flux integral

Let

$$\mathbf{F} = -3x\,\mathbf{i} - 4xy\,\mathbf{j}$$

then

$$\begin{aligned}
\oint_R \mathbf{F}\cdot\mathbf{n}\,ds &= \oint_C -3x\,dy - (-4xy)\,dx \\
&= \iint_R \left(\frac{\partial(-3x)}{\partial x} + \frac{\partial(-4xy)}{\partial y}\right)\,dA \\
&= \iint_R (-3 - 4x)\,dA
\end{aligned}$$

by Green's Theorem.

(b) $\iint_S \mathbf{G} \cdot \mathbf{n} \, dS$

This is the flux of the three-dimensional vector field $\mathbf{G}$ across the oriented surface S. It is also just the surface integral of the real-valued function $\mathbf{G} \cdot \mathbf{n}$.

(c) $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, dS$

This is a “flux of the curl” integral. If ∂S is nice enough, then we may apply Stokes’ Theorem to conclude that this is also a circulation integral

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, dS = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

Example 5. Let $\mathbf{F} = x^2y\,\mathbf{i} + 2y^3z\,\mathbf{j} + 3z\,\mathbf{k}$ and let S be the surface whose vector equation is

$$\mathbf{r}(s,t) = s\cos t\,\mathbf{i} + s\sin t\,\mathbf{j} + s\,\mathbf{k}, \quad (s,t) \in R$$

where $R = \{(s,t) \mid 0 \le s \le 1,\ 0 \le t \le 2\pi\}$. Calculate the flux of the curl of $\mathbf{F}$ across S in the direction away from the z-axis. So the vector normal to the surface should have a negative $\mathbf{k}$ component.

We remark that S is the cone $z = \sqrt{x^2+y^2},\ z \le 1$.

We first calculate $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\,dS$ directly. Proceeding in the usual way we have

$$\mathbf{r}_s = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \mathbf{k}$$

$$\mathbf{r}_t = -s\sin t\,\mathbf{i} + s\cos t\,\mathbf{j}$$

Thus

$$\mathbf{r}_s \times \mathbf{r}_t = -s\cos t\,\mathbf{i} - s\sin t\,\mathbf{j} + s\,\mathbf{k}$$

Since the $\mathbf{k}$ component is positive, we choose

$$\mathbf{r}_t \times \mathbf{r}_s = -\mathbf{r}_s \times \mathbf{r}_t = s\cos t\,\mathbf{i} + s\sin t\,\mathbf{j} - s\,\mathbf{k}$$

A routine calculation yields

$$\begin{aligned}\nabla \times \mathbf{F} &= -2y^3\,\mathbf{i} - x^2\,\mathbf{k}\\ &= -2s^3\sin^3 t\,\mathbf{i} - s^2\cos^2 t\,\mathbf{k}\end{aligned}$$

So that

$$\begin{aligned}\nabla \times \mathbf{F} \cdot (\mathbf{r}_t \times \mathbf{r}_s) &= (-2s^3\sin^3 t\,\mathbf{i} - s^2\cos^2 t\,\mathbf{k}) \cdot (s\cos t\,\mathbf{i} + s\sin t\,\mathbf{j} - s\,\mathbf{k})\\ &= s^3\cos^2 t - 2s^4\sin^3 t\cos t\end{aligned}$$

It follows that

$$\begin{aligned}\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_R \nabla \times \mathbf{F} \cdot (\mathbf{r}_t \times \mathbf{r}_s)\ dA \\ &= \iint_R \left(s^3 \cos^2 t - 2s^4 \sin^3 t \cos t\right)\ dA \\ &= \iint_R s^3 \cos^2 t\, dA - \iint_R 2s^4 \sin^3 t \cos t\, dA\end{aligned}$$

Now the second integral is zero since

$$\begin{aligned}\iint_R 2s^4 \sin^3 t \cos t\, dA &= \int_0^{2\pi}\int_0^1 2s^4 \sin^3 t \cos t\, ds\, dt \\ &= \int_0^1 2s^4\, ds \int_0^{2\pi} \sin^3 t \cos t\, dt \\ &= 0\end{aligned}$$

For the first integral we have

$$\begin{aligned}\iint_R s^3 \cos^2 t\, dA &= \int_0^1 s^3\, ds \int_0^{2\pi} \cos^2 t\, dt \\ &= \frac{1}{4} \times \pi\end{aligned}$$

It follows that

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS = \frac{\pi}{4} - 0$$

Example 6. Rework the previous example by evaluating the integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly and applying Stokes' Theorem. Here C is the boundary of the surface S from Example 5.

Notice that C is circle $x^2 + y^2 = 1,\ z = 1$. Thus C can be parameterized by the vector equation

$$\mathbf{r}(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \mathbf{k}, \quad 0 \le t \le 2\pi$$

since the circle must be parameterized in the clockwise direction when viewed from above. It follows that

$$\frac{d\mathbf{r}}{dt} = -\sin t\,\mathbf{i} - \cos t\,\mathbf{j}$$

and

$$\mathbf{F}(\mathbf{r}(t)) = -\cos^2 t \sin t\,\mathbf{i} - 2\sin^3 t\,\mathbf{j} + 3\,\mathbf{k}$$

Thus

$$\mathbf{F} \cdot d\mathbf{r} = \left(\cos^2 t \sin^2 t + 2\sin^3 t \cos t\right) dt$$

So by Stokes' theorem

$$\begin{aligned}\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS &= \oint_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^{2\pi} \left(\cos^2 t \sin^2 t + 2\sin^3 t \cos t\right) dt \\ &= \int_0^{2\pi} \cos^2 t \sin^2 t\, dt + 0 \\ &= \frac{1}{4}\int_0^{2\pi} \sin^2 2t\, dt \\ &= \frac{\pi}{4}\end{aligned}$$

as we saw above.

Example 7. Let $\mathbf{F} = y\,\mathbf{i} + xz\,\mathbf{j} + x^2\,\mathbf{k}$ and let C be the boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant.

Calculate the circulation of $\mathbf{F}$ around C counterclockwise when viewed from above. That is, evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

Let S be the given triangular region. Then S can be parameterized by the vector equation

$$\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + (1 - x - y)\,\mathbf{k}, \quad (x, y) \in R$$

where $R = \{(x, y) \mid 0 \le x \le 1,\ 0 \le y \le 1 - x\}$. Thus

$$\mathbf{r}_x \times \mathbf{r}_y = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

and

$$\nabla \times \mathbf{F} = -x\,\mathbf{i} - 2x\,\mathbf{j} - (x + y)\,\mathbf{k}$$

Then by Stokes' Theorem

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS \\
&= \iint_R (-x\,\mathbf{i} - 2x\,\mathbf{j} - (x + y)\,\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})\ dA \\
&= -\int_0^1 \int_0^{1-x} (4x + y)\, dy\, dx \\
&= \vdots \\
&= -5/6
\end{aligned}$$

Example 8. Rework the previous example by evaluating the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly.

Let C_1 be the line segment from $P(1,0,0)$ to $Q(0,1,0)$. Then C_1 can be parameterized by the vector equation

$$\mathbf{r}(t) = (1-t)\,\mathbf{i} + t\,\mathbf{j}, \quad t \in [0,1]$$

Then $d\mathbf{r} = (-\,\mathbf{i} + \,\mathbf{j})\,dt$ and

$$\mathbf{F}(\mathbf{r}(t)) = t\,\mathbf{i} + (1-t)^2\,\mathbf{k} \quad \text{and} \quad \mathbf{F} \cdot d\mathbf{r} = -t\,dt$$

Thus

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = -\int_0^1 t\,dt = -1/2$$

Now let C_2 be the line segment from Q to $T(0,0,1)$. It is straightforward to show that $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0$. Finally, let C_3 be the line segment from T to P. Then C_3 can be parameterized by the vector equation

$$\mathbf{r}(t) = t\,\mathbf{j} + (1-t)\,\mathbf{k}, \quad t \in [0,1]$$

So that

$$d\mathbf{r} = (\,\mathbf{j} - \,\mathbf{k})\,dt \quad \text{and} \quad \mathbf{F} \cdot d\mathbf{r} = -t^2\,dt$$

It follows that

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 -t^2\,dt = -1/3$$

Thus

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r}$$
$$= -1/2 + 0 - 1/3 = -5/6$$

as we saw above.

Example 9. Let S be the cylinder $x^2 + y^2 = a^2,\ 0 \le z \le h$, together with its top, $x^2 + y^2 \le a^2,\ z = h$. Let $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j} + x^2\,\mathbf{k}$. Calculate the flux of $\nabla \times \mathbf{F}$ outward through S.

The boundary of S has the vector equation

$$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}, \quad 0 \le t \le 2\pi$$

Now

$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j}, \\ \mathbf{F}(\mathbf{r}(t)) &= -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} \end{aligned}$$

So by Stokes' Theorem

$$\begin{aligned} \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\,dA &= \oint_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^{2\pi} a^2(\sin^2 t + \cos^2 t)\,dt = 2\pi a^2 \end{aligned}$$

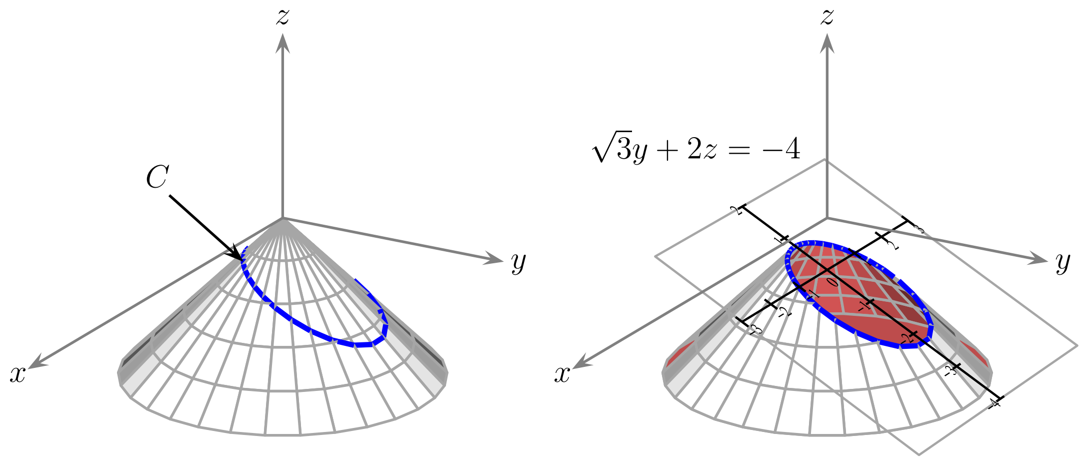

Figure 4: Space curve generated by the intersection of a plane with an inverted cone.

Example 10. Let $\mathbf{F} = \langle -6y, y^2 z, 2x \rangle$ and let C be the closed curve generated by the intersection of the cone $z = -\sqrt{x^2 + y^2}$ and the plane $\sqrt{3}y + 2z = -4$. The curve C (an ellipse) is shown in Figure 4. Evaluate the circulation integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$. Orient C to be counterclockwise when viewed from above. (C.f. Example 1 from the text book.)

Instead of evaluating the integral directly, let's appeal to Stokes' Theorem. A straightforward calculation yields curl $\mathbf{F} = \nabla \times \mathbf{F} = -y^2\,\mathbf{i} - 2\,\mathbf{j} + 6\,\mathbf{k}$. Although we are free to choose any (nice) surface whose boundary is C, it will be easiest if we work with the elliptical region S in the plane $\sqrt{3}y + 2z = -4$ that is bounded by C. Figure 5 shows the surface S reflected across the xy-plane (for easier viewing).

Now let R_{xy} be the projection of S onto the xy-plane. We leave it as an exercise to show that the boundary of R_{xy} is the ellipse

(3) $$g(x, y) \stackrel{\text{def}}{=} \frac{x^2}{16} + \frac{(y - 4\sqrt{3})^2}{64} = 1$$

So the region R_{xy} is defined by the inequality $g(x, y) \leq 1$.

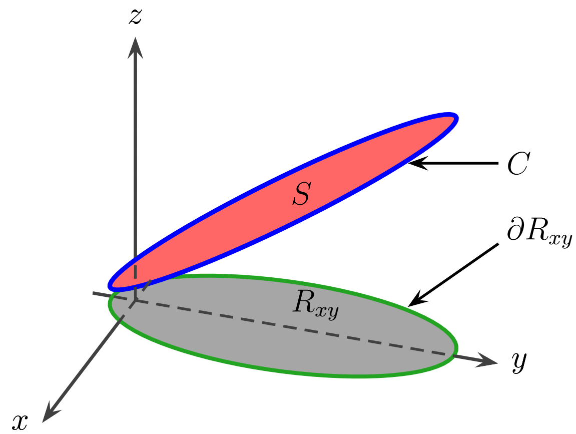

Figure 5: (Reflected) Surface S and its projection R_{xy}

Notice that we can parameterize S by the vector equation

(4) $$\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} - \underbrace{\left(2 + \frac{\sqrt{3}y}{2}\right)}_{h}\mathbf{k}, \quad (x,y) \in R_{xy}$$

It follows that

$$\mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial h}{\partial x}\mathbf{i} - \frac{\partial h}{\partial y}\mathbf{j} + \mathbf{k} = \frac{\sqrt{3}}{2}\mathbf{j} + \mathbf{k}$$

So by Stokes' Theorem

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,dS\\ &= \iint_{R_{xy}} (-y^2\,\mathbf{i} - 2\,\mathbf{j} + 6\,\mathbf{k})\cdot\left(\frac{\sqrt{3}}{2}\mathbf{j} + \mathbf{k}\right)\,dA\\ &= (6-\sqrt{3})\iint_{R_{xy}} dA\\ &= (6-\sqrt{3})\times \text{area of the ellipse from (3)}\\ &= (6-\sqrt{3})\times 32\pi\end{aligned}$$

Example 11. Redo the previous example by directly computing the circulation integral $\int_C \mathbf{F}\cdot d\mathbf{r}$. Notice that C can be defined by the vector equation

$$\mathbf{r}(t) = 4\cos t\,\mathbf{i} + (8\sin t + 4\sqrt{3})\,\mathbf{j} - (8 + 4\sqrt{3}\sin t)\,\mathbf{k}, \quad 0\le t\le 2\pi$$

and

$$d\mathbf{r} = (-4\sin t\,\mathbf{i} + 8\cos t\,\mathbf{j} - 4\sqrt{3}\cos t\,\mathbf{k})\,dt$$

Thus

$$\mathbf{F} = -6(8\sin t + 4\sqrt{3})\,\mathbf{i} + (8\sin t + 4\sqrt{3})^2(8 + 4\sqrt{3}\sin t)\,\mathbf{j} + 8\cos t\,\mathbf{k}$$

so that

$$\begin{aligned}\mathbf{F}\cdot d\mathbf{r} = (&24\sin t(8\sin t + 4\sqrt{3})\\ &+ 8\cos t(8\sin t + 4\sqrt{3})^2(8 + 4\sqrt{3}\sin t) - 32\sqrt{3}\cos^2 t)\,dt\end{aligned}$$

It follows that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F} \cdot d\mathbf{r} = \cdots = 32(6 - \sqrt{3})\pi$$

as we saw above. However, computing the surface integral was certainly easier than the calculation above.

Example 12. Redo Example 10 by using the vector equation below instead of (4).

(5)
$$\mathbf{r}(x, z) = x\,\mathbf{i} + \underbrace{\frac{-2}{\sqrt{3}}(2 + z)}_{q}\,\mathbf{j} + z\,\mathbf{k}, \quad (x, z) \in R_{xz}$$

Then

$$\mathbf{r}_x \times \mathbf{r}_z = -\frac{\partial q}{\partial x}\mathbf{i} + \mathbf{j} - \frac{\partial q}{\partial z}\mathbf{j} = \mathbf{j} + \frac{2}{\sqrt{3}}\mathbf{k}$$

Here R_{xz} is the projection of S onto the xz-plane. We leave it as an exercise to show that R_{xz} is the set of points (x, z) that satisfy

$$\frac{x^2}{16} + \frac{(z + 8)^2}{48} \leq 1$$

Once again by Stokes' Theorem we have

$$
\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,dS \\
&= \iint_{R_{xz}} (-y^2\,\mathbf{i} - 2\,\mathbf{j} + 6\,\mathbf{k})\cdot\left(\mathbf{j} + \frac{2}{\sqrt{3}}\,\mathbf{k}\right)\,dA \\
&= \left(\frac{12}{\sqrt{3}} - 2\right)\iint_{R_{xz}} dA \\
&= \left(\frac{12}{\sqrt{3}} - 2\right)\times \text{area of the elliptical region } R_{xz} \\
&= \left(\frac{12}{\sqrt{3}} - 2\right)\times 16\pi\sqrt{3} \\
&= 32(6-\sqrt{3})\pi
\end{aligned}
$$

Example 13. Now let $\mathbf{G} = \left\langle x^2y, \dfrac{x^3}{3}, \dfrac{-2xy}{\sqrt{3}} \right\rangle$ and let C be the ellipse defined in Example 10. Find $\oint_C \mathbf{G} \cdot d\mathbf{r}$.

Let S and R_{xy} be as defined in Example 10. Then by Stokes' Theorem

$$\begin{aligned}
\oint_C \mathbf{G} \cdot d\mathbf{r} &= \iint_S \nabla \times \mathbf{G} \cdot \mathbf{n}\, dS \\
&= \frac{1}{\sqrt{3}} \iint_{R_{xy}} (-2x\,\mathbf{i} + 2y\,\mathbf{j}) \cdot \left(\frac{\sqrt{3}}{2}\mathbf{j} + \mathbf{k} \right) dA \\
&= \iint_{R_{xy}} y\, dA \\
&= \int_{-8+4\sqrt{3}}^{8+4\sqrt{3}} \int_{\frac{-\sqrt{64-(y-4\sqrt{3})^2}}{2}}^{\frac{\sqrt{64-(y-4\sqrt{3})^2}}{2}} y\, dx\, dy \\
&= \int_{-8+4\sqrt{3}}^{8+4\sqrt{3}} y\sqrt{64 - (y - 4\sqrt{3})^2}\, dy \\
&= \int_{-8}^{8} (w + 4\sqrt{3})\sqrt{64 - w^2}\, dw \\
&= \int_{-8}^{8} w\sqrt{64 - w^2}\, dw + 4\sqrt{3} \int_{-8}^{8} \sqrt{64 - w^2}\, dw \\
&= 0 + 4\sqrt{3} \times 32\pi
\end{aligned}$$

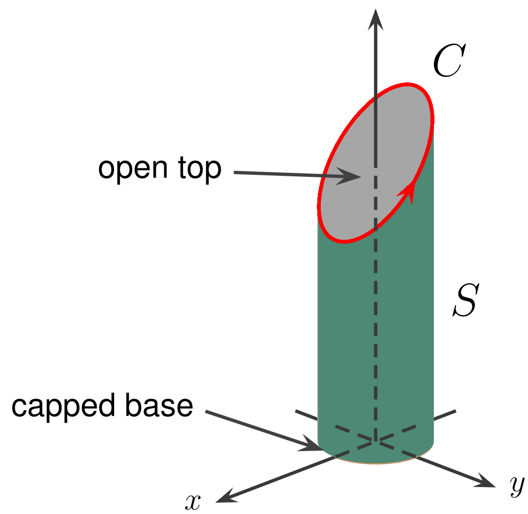

Figure 6: Cylindrical surface S bounded by the space curve C

Example 14. Let $\mathbf{F} = \langle 2z, 3xy^2, x^2 + y \rangle$ and let S be the cylindrical shell $x^2 + y^2 = 1$, bounded below by the unit disk $x^2 + y^2 \leq 1$ in the xy-plane and with an open top that lies in the plane $T : 2x + z = 6$. Also, let C be the boundary of S oriented counterclockwise when viewed from above (see Figure 6). Use Stokes' Theorem to evaluate the circulation of $\mathbf{F}$ around C.

As a general rule, we are permitted to use the most convenient piecewise smooth surface that has boundary C (which would be the ellipse in the plane T in this case). However, the theorem must also hold for S. Notice that $S = S_B \cup S_C$ where S_B is the base (in the xy-plane and S_C is the cylinder.

A routine calculation yields $\nabla \times \mathbf{F} = \langle 1, 2 - 2x, 3y^2 \rangle$.

For the base we have

$$\iint_{S_B} \nabla \times F \cdot \mathbf{n}\, dS = \iint_{S_B} \nabla \times F \cdot \ \mathbf{k}\, dS$$
$$= \iint_{\text{unit disk}} (3y^2 - 0)\, dA$$
$$= 3 \int_0^{2\pi} \int_0^1 r^2 \sin^2 \theta\, r\, dr\, d\theta$$
$$= \frac{3}{4} \int_0^{2\pi} \sin^2 \theta\, d\theta = \frac{3\pi}{4}$$

Notice that S_C can be parameterized by the vector equation

$$\mathbf{r}(s,t) = \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + s\, \mathbf{k}, \quad (s,t) \in D$$

where $D = \{(s,t) : 0 \le t \le 2\pi,\ 0 \le s \le 6 - 2\cos t\}$. Also,

$$\mathbf{r}_s \times \mathbf{r}_t = -\cos t\, \mathbf{i} - \sin t\, \mathbf{j}$$

Now

$$\iint_{S_C} \nabla \times F \cdot \mathbf{n}\, dS = \int_0^{2\pi} \int_0^{6-2\cos t} \left\langle 1, 2 - 2\cos t, 3\sin^2 t \right\rangle \cdot \langle -\cos t, -\sin t \rangle\, ds\, dt$$
$$= \int_0^{2\pi} \int_0^{6-2\cos t} (2\cos t - 2)\sin t - \cos t\, ds\, dt$$
$$\vdots$$
$$= 2\pi$$

Thus

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{n}\, dS = \frac{3\pi}{4} + 2\pi$$

Exercise: In Figure 6 from Example 14, let S_E be the elliptical region that lies in the plane T (and is bounded by C). Find the flux of the curl $\iint_{S_E} \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS$. Also, evaluate the circulation integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly.

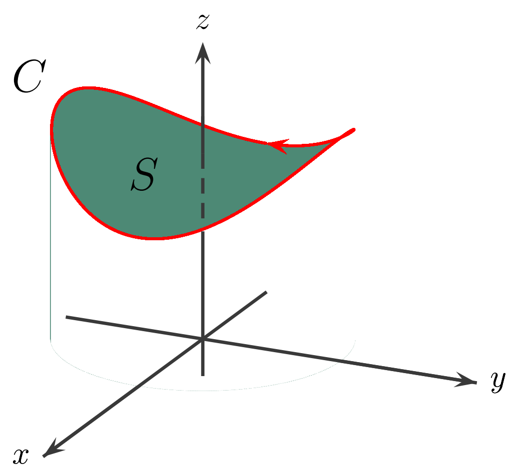

Figure 7: Curved elliptical surface S bounded by the space curve C

Example 15. Let $\mathbf{F} = \langle -3y, x^2 z, x\rangle$ and let C be the intersection of the cylinders $x^2 + y^2 = 4$ and $z = 4 + y^2/2$. Find the counterclockwise circulation (when viewed from above) of $\mathbf{F}$ around C (see Figure 7).

Let S be the interior of C on the cylinder $z = 4 + y^2/2$. Then S can be defined by the vector equation

$$\mathbf{r}(s,t) = \left\langle s\cos t, s\sin t, 4 + \frac{s^2 \sin^2 t}{2} \right\rangle, \ (s,t) \in D$$

where $D = \{(s,t)\,|\,0 \le s \le 2,\ 0 \le t \le 2\pi\}$. Now

$$\mathbf{r}_s = \langle \cos t, \sin t, s\sin^2 t\rangle$$

$$\mathbf{r}_t = \langle -s\sin t, s\cos t, s^2 \sin t\ \cos t\rangle$$

so that

$$\mathbf{r}_s \times \mathbf{r}_t = \langle 0, -s^2 \sin t, s\rangle$$

Now a routine calculation shows that $\nabla \times \mathbf{F} = \left\langle -x^2, -1, 3+2xz \right\rangle$. Thus

$$\nabla \times \mathbf{F}(\mathbf{r}(s,t)) = \left\langle -s^2\cos^2 t, -1, 3+2s\cos t\left(4+\frac{s^2\sin^2 t}{2}\right)\right\rangle$$

and

$$\nabla \times \mathbf{F}(\mathbf{r}(s,t)) \cdot (\mathbf{r}_s \times \mathbf{r}_t) = 3s + 8s^2\cos t + s^2\sin t + s^4\cos t\,\sin^2 t$$

So by Stokes' Theorem

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \nabla \times F \cdot \mathbf{n}\, dS \\
&= \iint_D \nabla \times \mathbf{F}(\mathbf{r}(s,t)) \cdot (\mathbf{r}_s \times \mathbf{r}_t)\, ds\, dt \\
&= \int_0^{2\pi}\int_0^2 (3s + 8s^2\cos t + s^2\sin t + s^4\cos t\,\sin^2 t)\, ds\, dt \\
&= 6\pi \int_0^2 s\, ds + 0 + 0 + 0 \\
&= 12\pi
\end{aligned}$$

Exercise: In the above example, evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ directly.

Integral Theorems, Flux and Flow - Summary

As we saw earlier, we can imagine the **del** operator defined in this chapter as also being defined on two-dimensional vector fields by writing

$$\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} = M\,\mathbf{i} + N\,\mathbf{j} + 0\,\mathbf{k}$$

whenever it is appropriate.

Now using the “del” notation we can rewrite all the integral theorems using a *uniform* notation.

We recall a few important definitions.

Definition. Circulation Density at a Point

The **circulation density** or **curl** of a vector field $\mathbf{F}$ is

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F}$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

As we saw earlier, this reduces to the usual $\mathbf{k}$-component of curl whenever $P = 0$ and $\mathbf{F} = \mathbf{F}\Big|_{z=0}$.

Definition. Flux Density at a Point

The **flux density** or **divergence** of a vector field $\mathbf{F}$ is

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F}$$

Once again, this reduces to the usual two-dimensional version whenever $P = 0$.

For circulation around a smooth closed curve C we have

Green's Theorem (Tangential Form):

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C M\,dx + N\,dy$$
$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$
$$= \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\,dA$$

Stokes' Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\,dS$$

In the first case, C is the boundary of the plane region R. In the second, C is the boundary of the oriented surface S.

For the flux around the smooth closed curve C of an orientable surface S we have

Green's Theorem (Normal Form):

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \oint_C M \, dy - N \, dx$$
$$= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \, dA$$
$$= \iint_R \nabla \cdot \mathbf{F} \, dA$$

Divergence Theorem:

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_D \nabla \cdot \mathbf{F} \, dV$$

Once again C is the boundary of the plane region R and D is the region enclosed by the oriented surface S.