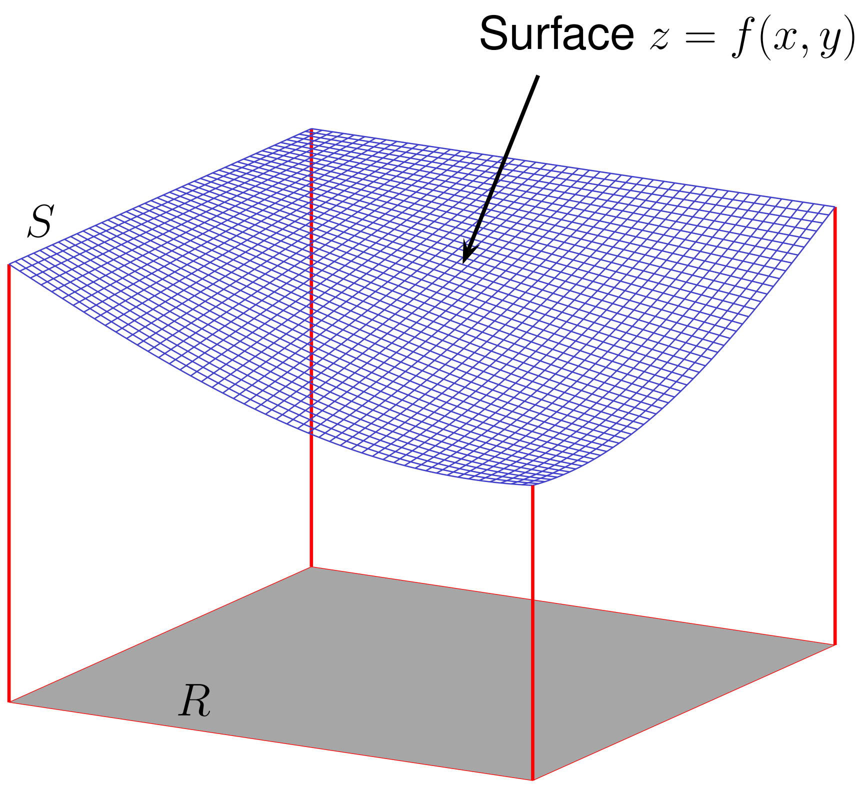
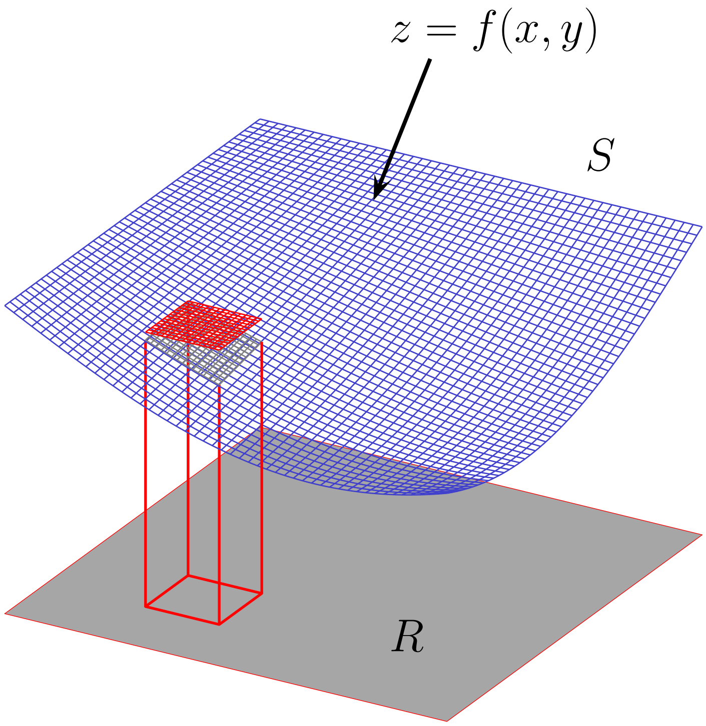
[Video Link](#)

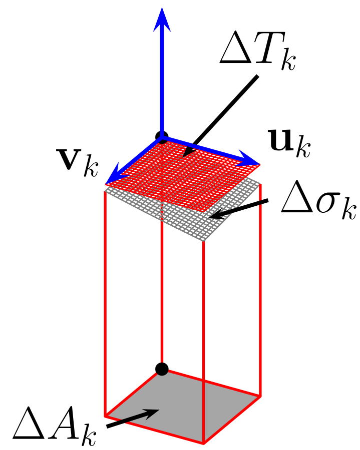
15.6 Surface Area and Surface Integrals

Surface Area



Suppose that we wish to measure the “area” of the level surface $z = f(x, y)$. Let S denote this area and let R be the “shadow” of S on one of the coordinate planes. As usual, we partition the region R into small rectangles each of area ΔA_k (see sketch below).





Now let $\Delta\sigma_k$ denote the area of the surface directly above the region ΔA_k and let ΔT_k denote the area of the corresponding parallelogram on the tangent plane at the point of tangency, say P_k. If ΔA_k is small then $\Delta T_k \approx \Delta\sigma_k$. Thus

$$\sum \Delta\sigma_k \approx \sum \Delta T_k$$

so that

$$\text{Area}(S) = \lim \sum \Delta T_k$$

So what is ΔT_k. Let $\mathbf{u}_k$ and $\mathbf{v}_k$ be the vectors that correspond to the sides of the parallelogram with area ΔT_k. In section 12.4 we saw that $\Delta T_k = |\mathbf{u}_k \times \mathbf{v}_k|$.

Suppose that $P_k = P_k(x_k, y_k, f(x_k, y_k))$ and recall from section 14.3 that

$$\mathbf{v}_k = \Delta x\,\mathbf{i} + f_x(x_k, y_k)\,\Delta x\,\mathbf{k}$$
$$\mathbf{u}_k = \Delta y\,\mathbf{j} + f_y(x_k, y_k)\,\Delta y\,\mathbf{k}$$

Then

$$\begin{aligned}
\mathbf{v}_k \times \mathbf{u}_k &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & f_x(x_k, y_k)\,\Delta x \\ 0 & \Delta y & f_y(x_k, y_k)\,\Delta y \end{vmatrix} \\
&= \mathbf{i} \begin{vmatrix} 0 & f_x(x_k, y_k)\,\Delta x \\ \Delta y & f_y(x_k, y_k)\,\Delta y \end{vmatrix} - \mathbf{j} \begin{vmatrix} \Delta x & f_x(x_k, y_k)\,\Delta x \\ 0 & f_y(x_k, y_k)\,\Delta y \end{vmatrix} + \mathbf{k} \begin{vmatrix} \Delta x & 0 \\ 0 & \Delta y \end{vmatrix} \\
&= \vdots \\
&= [-f_x(x_k, y_k)\,\mathbf{i} - f_y(x_k, y_k)\,\mathbf{j} + \mathbf{k}]\,\underbrace{\Delta x\,\Delta y}_{\Delta A}
\end{aligned}$$

So that

$$\begin{aligned}
\Delta T_k &= |\mathbf{u}_k \times \mathbf{v}_k| \\
&= \sqrt{[f_x(x_k, y_k)]^2 + [f_y(x_k, y_k)]^2 + 1}\,\Delta A
\end{aligned}$$

It follows that the surface area is given by

$$\begin{aligned}\text{Area}(S) &= \lim \sum \Delta T_k \\ &= \lim \sum \sqrt{1 + [f_x(x_k, y_k)]^2 + [f_y(x_k, y_k)]^2}\, \Delta A \\ &= \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA\end{aligned}$$

provided the limit exists. This leads to the following definition.

Definition. The Formula for Surface Area

The area of the surface S defined by $z = f(x, y)$ over a closed and bounded plane region R, with continuous partials f_x and f_y, is given by

$$\text{Area}(S) = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA$$

or, alternatively,

$$\text{Area}(S) = \iint_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA$$

Remark. It is worth mentioning that the surface S can also be realized as the level surface $F(x, y, z) = z - f(x, y)$. Now observe that

$$\nabla F = -f_x\,\mathbf{i} - f_y\,\mathbf{j} + \mathbf{k}$$

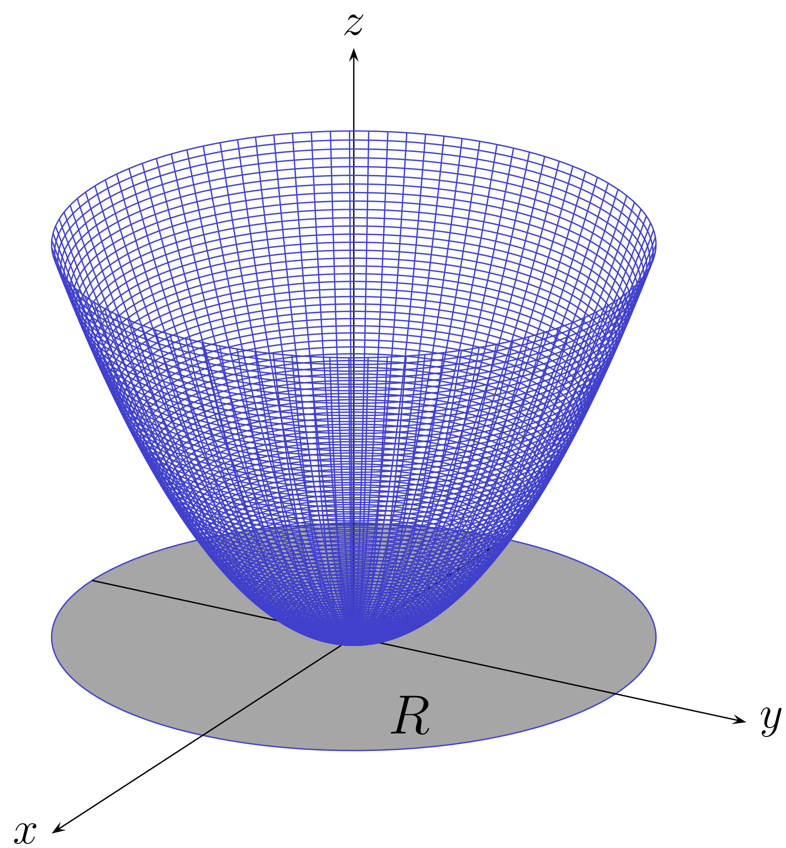
so that

$$|\nabla F| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

Example 1. Finding Surface Area

Find the area of the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

The surface S along with the region R are shown in the sketch below.



Thus

$$\frac{\partial z}{\partial x} = 2x \quad \text{and} \quad \frac{\partial z}{\partial y} = 2y$$

so that

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + 4x^2 + 4y^2}$$

It follows that the surface area is

$$\begin{aligned}
\text{A}(S) &= \iint_R \sqrt{1 + 4x^2 + 4y^2}\, dA \\
&= \iint_{x^2+y^2 \le 4} \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy \\
&= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta \\
&= 2\pi \int_0^2 \sqrt{1 + 4r^2}\, r\, dr \\
&= \frac{2\pi}{12} (1 + 4r^2)^{3/2} \Big|_0^2 \\
&= \frac{\pi}{6} \left(17\sqrt{17} - 1\right)
\end{aligned}$$

16.6 Parametric Surfaces and Their Areas

Parametric Surfaces

In this section we study the vector valued function $\mathbf{r}(u, v)$ of two parameters u and v. So let

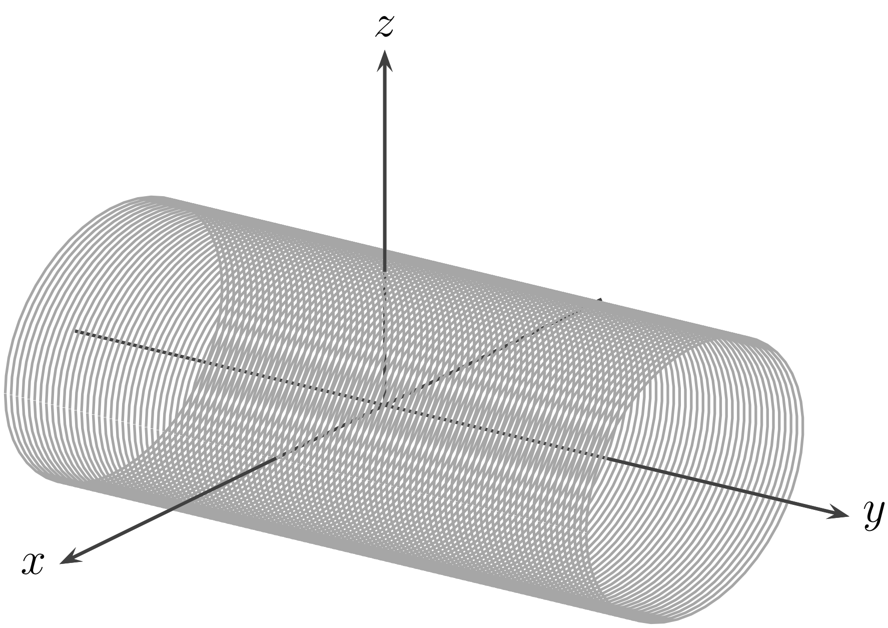
(1) $$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

defined on a region D of the so-called uv-plane.

The set of points $(x, y, z) \in \mathbb{R}^3$ with

(2) $$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad (u, v) \in D$$

is called a **parametric surface** S and the equations (2) are called the parametric equations of S.



Example 2. Identify and sketch the surface whose vector equation is

$$\mathbf{r}(u, v) = \cos u\,\mathbf{i} + v\,\mathbf{j} + \frac{3\sin u}{4}\,\mathbf{k}$$

The corresponding parametric equations are

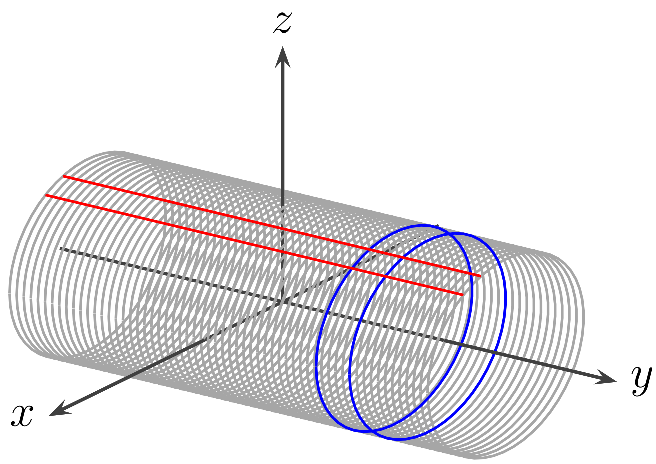
$$x = \cos u, \quad y = v, \quad z = \frac{3\sin u}{4}$$

Notice that

$$9x^2 + 16z^2 = 9\cos^2 u + 9\sin^2 u = 9$$

So that cross-sections parallel to the xz-plane are ellipses. Since $y = v$ without restriction, we obtain an elliptical cylinder parallel to the y-axis.

Suppose now that we fix $u = u_0$. Then $\mathbf{r}_1(v) = \mathbf{r}(u_0, v)$ is a vector-valued function of a single parameter v. Similarly, $\mathbf{r}_2(u) = \mathbf{r}(u, v_0)$ is a vector-valued function of the single parameter u. In each case, we generate families of *space curves* that lie on the surface S. A few of these surface curves are shown on the surface below (from the previous example).



It turns out to be very straightforward to find the parametric representation for a given surface of the form $z = f(x, y)$.

Example 3. Find the parametric representation of the paraboloid $z = x^2 + y^2 + 1$.

We give two representations.

The Easy One: Here we let $x = x$ and $y = y$. Then $z = x^2 + y^2 + 1$ so that

$$\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + (x^2 + y^2 + 1)\,\mathbf{k}$$

The More Useful Representation (perhaps): For this one we work with the polar parameters r and θ. So let $x = r\cos\theta$ and $y = r\sin\theta$. It follows that $z = r^2 + 1$ so that

$$\mathbf{r}(r, \theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + (r^2 + 1)\,\mathbf{k}$$

Parametric Surfaces and Tangent Planes

This is presents no difficulties. See the text.

Surface Area

It turns out the we can derive the formula for the area of parametric surface using a similar approach to the one used above. The details are outlined in the text. We have

Definition. If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}, \quad (u,v) \in D$$

and S is covered just once as (u,v) ranges over the parameter domain D, then the surface area of S is given by

(3)
$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \; dA$$

where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\,\mathbf{i} + \frac{\partial y}{\partial u}\,\mathbf{j} + \frac{\partial z}{\partial u}\,\mathbf{k}$$
$$\mathbf{r}_v = \frac{\partial x}{\partial v}\,\mathbf{i} + \frac{\partial y}{\partial v}\,\mathbf{j} + \frac{\partial z}{\partial v}\,\mathbf{k}$$

Example 4. Find the area of the helicoid (see Figure 1) whose vector equation is

(4) $$\mathbf{r}(s,t) = s\cos t\,\mathbf{i} + s\sin t\,\mathbf{j} + t\,\mathbf{k}, \quad 0 \le s \le 1,\ 0 \le t \le \pi$$

We compute the first partials to obtain

$$\mathbf{r}_s = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$$
$$\mathbf{r}_t = -s\sin t\,\mathbf{i} + s\cos t\,\mathbf{j} + \mathbf{k}$$

So that

$$\mathbf{r}_s \times \mathbf{r}_t = \sin t\,\mathbf{i} - \cos t\,\mathbf{j} + s\,\mathbf{k}$$

and hence

$$|\mathbf{r}_s \times \mathbf{r}_t|^2 = 1 + s^2$$

So by (3) the surface area is

$$\begin{aligned} A &= \iint_D |\mathbf{r}_s \times \mathbf{r}_t|\, dA \\ &= \int_0^{\pi}\int_0^1 \sqrt{1+s^2}\,ds\,dt \\ &= \pi\int_0^1 \sqrt{1+s^2}\,ds \qquad (5) \\ &= \vdots \\ &= \pi\left(\frac{\sqrt{2} + \ln\left(\sqrt{2}+1\right)}{2}\right) \end{aligned}$$

Here we have suppressed the calculations involving trigonometric substitution and the subsequent integration by parts.

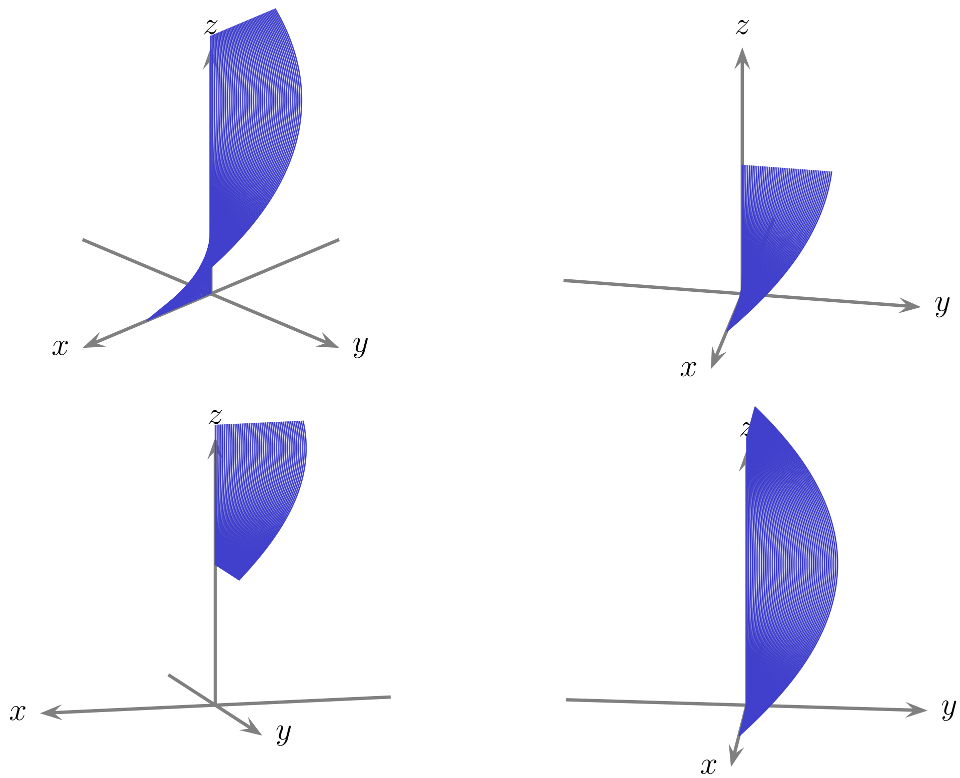


Figure 1: Several Views of the Helicoid

Example 5. Redo the previous example by writing the vector equation of the helicoid as $z = f(x, y)$ and using the parametric equation

(6) $$\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + f(x, y)\,\mathbf{k}$$

From equation (4) we let $x = s\cos t$, $y = s\sin t$ and let

$$R = \{(x, y) : -1 \le x \le 1, 0 \le y \le \sqrt{1 - x^2}\}$$

See Figure 2.

For $(x, y) \in R \setminus \{(0, 0)\}$ we let

$$z = t = f(x, y) = \begin{cases} \pi + \arctan(y/x) & \text{if } -1 \le x < 0 \\ \pi/2 & \text{if } x = 0 \\ \arctan(y/x) & \text{if } 0 < x \le 1 \end{cases}$$

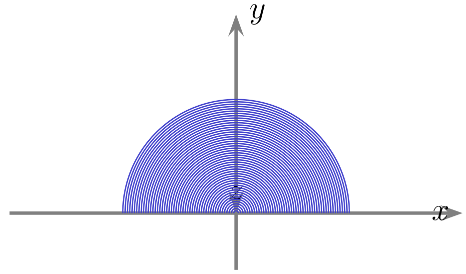


Figure 2: Bird's-eye View of the Helicoid

Rather than deal with the complexities of f along the y-axis, we will compute the surface of the portion of $z = f(x, y)$ that lies in the first quadrant and then attempt to exploit symmetry. So let $R^+ = R\Big|_{x>0}$ and

$$\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + \arctan\frac{y}{x}\ \mathbf{k},\ (x, y) \in R^+$$

Then

$$\begin{aligned}\mathbf{r}_x \times \mathbf{r}_y &= (\,\mathbf{i} + \frac{-y}{x^2+y^2}\,\mathbf{k}) \times (\,\mathbf{j} + \frac{x}{x^2+y^2}\,\mathbf{k})\\ &= \frac{y}{x^2+y^2}\,\mathbf{i} - \frac{x}{x^2+y^2}\,\mathbf{j} + \ \mathbf{k}\end{aligned}$$

It follows by (3) that surface area that lies in the first octant is

$$\begin{aligned}\mathsf{A}^+ &= \iint_{R^+} \sqrt{1 + \frac{x^2}{(x^2+y^2)^2} + \frac{y^2}{(x^2+y^2)^2}}\,dA\\ &= \iint_{R^+} \sqrt{1 + \frac{x^2+y^2}{(x^2+y^2)^2}}\,dA\\ &= \iint_{R^+} \sqrt{1 + \frac{1}{x^2+y^2}}\,dA\end{aligned}$$

Unfortunately, the integral is improper because of the integrand is unbounded at the origin.

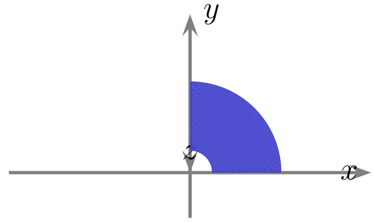


Figure 3: The Region R'

Now consider the region (see Figure 3)

$$R' = \{(r, \theta) : 0 < a \leq r \leq 1, 0 \leq \theta \leq \pi/2\}$$

The idea is that we can safely integrate over R' and then let a go to zero.

Switching to polar coordinates we obtain

$$\begin{aligned}\iint_{R'} \sqrt{1 + \frac{1}{x^2 + y^2}}\, dA &= \int_0^{\pi/2} \int_a^1 \sqrt{1 + \frac{1}{r^2}}\, r\, dr\, d\theta \\ &= \frac{\pi}{2} \int_a^1 \sqrt{1 + r^2}\, dr\end{aligned}$$

It follows that

$$\begin{aligned}\iint_{R^+} \sqrt{1 + \frac{1}{x^2 + y^2}}\, dA &= \frac{\pi}{2} \lim_{a \to 0^+} \int_a^1 \sqrt{1 + r^2}\, dr \\ &= \frac{\pi}{2} \int_0^1 \sqrt{1 + r^2}\, dr\end{aligned}$$

We leave it as an exercise to show that

$$\mathsf{A} = \pi \int_0^1 \sqrt{1 + r^2}\, dr$$

in agreement with (5).

Example 6. Find the surface area of the cylinder in Example 2 for $0 \le u \le 2\pi$ and $0 \le v \le 1$.

Once again, we compute the first partials to obtain

$$\mathbf{r}_u = -\sin u\,\mathbf{i} + \frac{3\cos u}{4}\,\mathbf{k}$$

$$\mathbf{r}_v = \mathbf{j}$$

So that

$$\mathbf{r}_u \times \mathbf{r}_v = \frac{-3\cos u}{4}\,\mathbf{i} - \sin u\,\mathbf{k}$$

and hence

$$|\mathbf{r}_u \times \mathbf{r}_v|^2 = \frac{9}{16}\cos^2 u + \sin^2 u$$

So by (3) the area is

$$\begin{aligned} A &= \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\; dA \\ &= \int_0^1 \int_0^{2\pi} \sqrt{\frac{9}{16}\cos^2 u + \sin^2 u}\, du\, dv \\ &= \int_0^{2\pi} \sqrt{\frac{9}{16}\cos^2 u + \sin^2 u}\, du \end{aligned}$$

Unfortunately, the last expression is an elliptical integral and cannot be evaluated by elementary methods. However, we can approximate the integral with the help of a CAS to obtain

$$A \approx 5.525873040$$

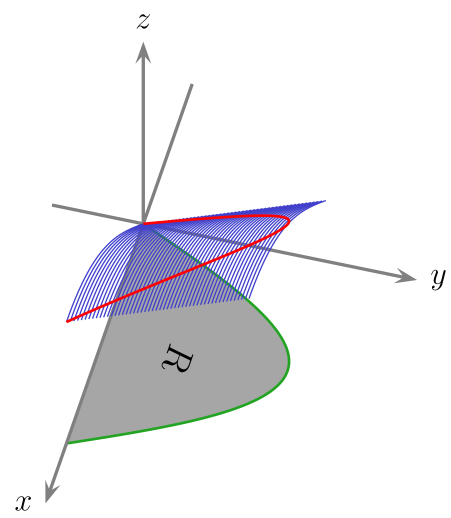


Figure 4: $z = y + 2\sin x$

Example 7. Find the area of surface $z = y + 2\sin x$ that lies above the region R in the xy-plane bounded by $y = 0,\ y = \sin 2x,\ 0 \le x \le \pi/2$. See Figure 4.

The given surface S can be defined by the vector equation

$$\mathbf{r}(x, y) = \langle x, y, y + 2\sin x\rangle\ ,\ (x, y) \in R$$

Now

$$\mathbf{r}_x = \langle 1, 0, 2\cos x\rangle$$

$$\mathbf{r}_y = \langle 0, 1, 1\rangle$$

and

$$\mathbf{r}_x \times \mathbf{r}_y = \langle -2\cos x, 1, 1\rangle$$

So that

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{2 + 4\cos^2 x}$$

$$= \sqrt{2 + 2(1 + \cos 2x)}$$

Thus

$$
\begin{aligned}
\text{area}(S) &= \iint_R \sqrt{4 + 2\cos 2x}\, dx\, dy \\
&= \int_0^{\pi/2} \int_0^{\sin 2x} \sqrt{4 + 2\cos 2x}\, dy\, dx \\
&= \int_0^{\pi/2} \sqrt{4 + 2\cos 2x}\, \sin 2x\, dx \\
&= \frac{1}{4} \int_2^6 \sqrt{u}\, du \\
&= \frac{6^{3/2} - 2^{3/2}}{6}
\end{aligned}
$$