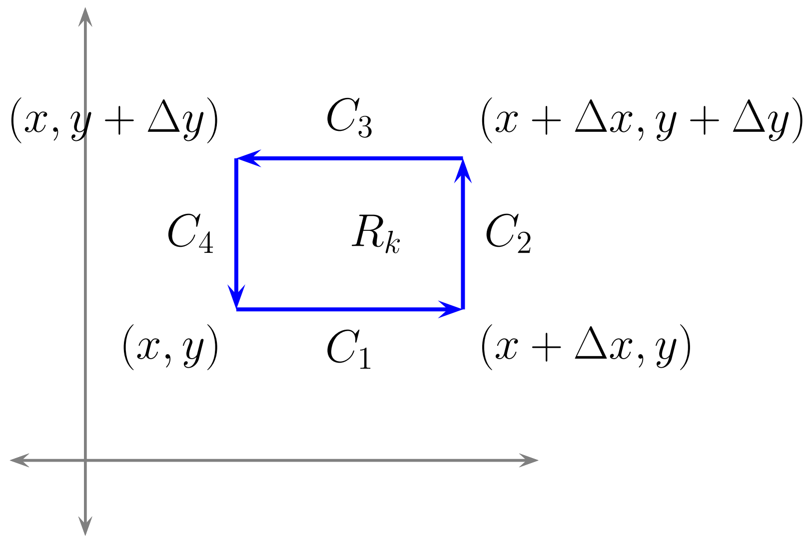
16.4 Green's Theorem

Circulation Density



Suppose that

$$\mathbf{F}(x,y) = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$$

is the velocity field of a fluid flow in the plane and that the first partials of M and N are continuous over a region R. Now let R_k be a small rectangle in R (as shown above).

Suppose that we wish to approximate the circulation around the small rectangle R_k. If Δx and Δy are small, we expect the velocity field to be nearly constant on each of the four sides of A. For example, the flow along C_1 is approximately

$$\mathbf{F}(x,y)\cdot \mathbf{i}\,\Delta x = M(x,y)\,\Delta x.$$

Similarly, the flow along each of the other three sides is

$$C_2\colon\ \mathbf{F}(x+\Delta x,y)\cdot \mathbf{j}\,\Delta y = N(x+\Delta x,y)\,\Delta y$$

$$C_3\colon\ \mathbf{F}(x,y+\Delta y)\cdot (-\,\mathbf{i})\,\Delta x = -M(x,y+\Delta y)\,\Delta x$$

$$C_4\colon\ \mathbf{F}(x,y)\cdot (-\,\mathbf{j})\,\Delta y = -N(x,y)\,\Delta y$$

Now putting the top and bottom sides together we have

$$\begin{aligned} M(x,y)\,\Delta x - M(x,y+\Delta y)\,\Delta x &= \left(\frac{M(x,y)-M(x,y+\Delta y)}{\Delta y}\right)\Delta x\,\Delta y \\ &\approx \frac{-\partial M}{\partial y}\,\Delta x\,\Delta y \end{aligned}$$

For the left and right sides we have

$$\begin{aligned} N(x+\Delta x,y)\,\Delta y - N(x,y)\,\Delta y &= \left(\frac{N(x+\Delta x,y)-N(x,y)}{\Delta x}\right)\Delta x\,\Delta y \\ &\approx \frac{\partial N}{\partial x}\,\Delta x\,\Delta y \end{aligned}$$

It follows that the circulation around the rectangle is approximately

(1) $$\text{circulation} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x\,\Delta y$$

Dividing both sides of (1) by the area of R_k suggests the following definition.

Definition. Circulation Density at a Point in the Plane

The **circulation density** or $\mathbf{k}$**-component of the curl** of a vector field $\mathbf{F}(x, y) = M\,\mathbf{i} + N\,\mathbf{j}$ at a point (x, y) is

$$(\text{curl}\,\mathbf{F}) \cdot \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \tag{2}$$

Remark. Recall that the curl of the vector field $\mathbf{F}$ is given by

$$\begin{aligned} \text{curl}\,\mathbf{F} &= \nabla \times \mathbf{F} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} \end{aligned}$$

Notice that circulation density in the plane is a **scalar**.

Theorem 1. Green's Theorem (Tangential Form)

Let C be a piecewise-smooth, simple closed curve in the plane and let R be the region bounded by C (in the plane). Suppose also that M and N have continuous partial derivatives on an open region that contains R.

Then the counterclockwise circulation of the field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ around C is equal to the double integral of the *circulation density* ($\mathbf{k}$-component of the curl) over the region R.

(3) $$\int_C \mathbf{F} \cdot \mathbf{T}\,ds = \oint_C M\,dx + N\,dy$$

(4) $$= \iint_R \underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{\text{circulation density}} \,dx\,dy$$

or, more conveniently,

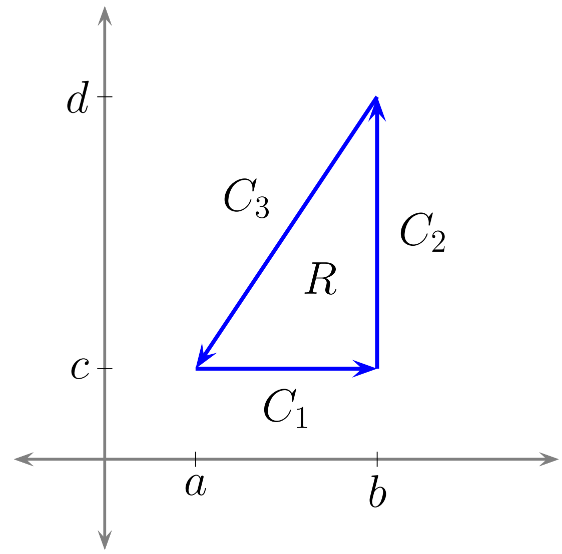
$$= \iint_R \underbrace{(\nabla \times \mathbf{F}) \cdot \mathbf{k}}_{\text{circulation density}} \,dx\,dy$$

Proof. We shall prove a special case of the theorem. Suppose that C is the boundary of the triangular region shown in the sketch.
Let

$$y = g(x) = \frac{d-c}{b-a}(x-a) + c \text{ and}$$

$$x = g^{-1}(y) = \frac{b-a}{d-c}(y-c) + c.$$

Notice that these are equations for C_3 and will be used below.



1. We start with the curl integral (RHS of (4)).

$$\iint_R \frac{\partial N}{\partial x}\,dx\,dy - \iint_R \frac{\partial M}{\partial y}\,dy\,dx$$

$$= \int_c^d \int_{x=g^{-1}(y)}^{x=b} \frac{\partial N}{\partial x}\,dx\,dy - \int_a^b \int_{y=c}^{y=g(x)} \frac{\partial M}{\partial y}\,dy\,dx$$

$$= \int_c^d \left(N(b,y) - N\left(g^{-1}(y),y\right)\right)\,dy$$

$$- \int_a^b \left(M\left(x,g(x)\right) - M(x,c)\right)\,dx$$

or

(5)
$$= \int_c^d \left(N(b,y) - N\left(g^{-1}(y),y\right)\right)\,dy$$

$$+ \int_a^b \left(M(x,c) - M\left(x,g(x)\right)\right)\,dx$$

2. Now we consider the circulation integral (RHS of (3)).

   It is easy to see that the circulation integral can be rewritten as

$$\oint_C M\,dx + N\,dy = \int_{C_1} M\,dx + \int_{C_2} N\,dy + \int_{C_3} M\,dx + \int_{C_3} N\,dy$$

   Now

$$\int_{C_1} M\,dx = \int_a^b M(x,c)\,dx$$

$$\int_{C_2} N\,dy = \int_c^d N(b,y)\,dy$$

$$\int_{C_3} M\,dx = \int_b^a M\left(x,g(x)\right)\,dx = -\int_a^b M\left(x,g(x)\right)\,dx$$

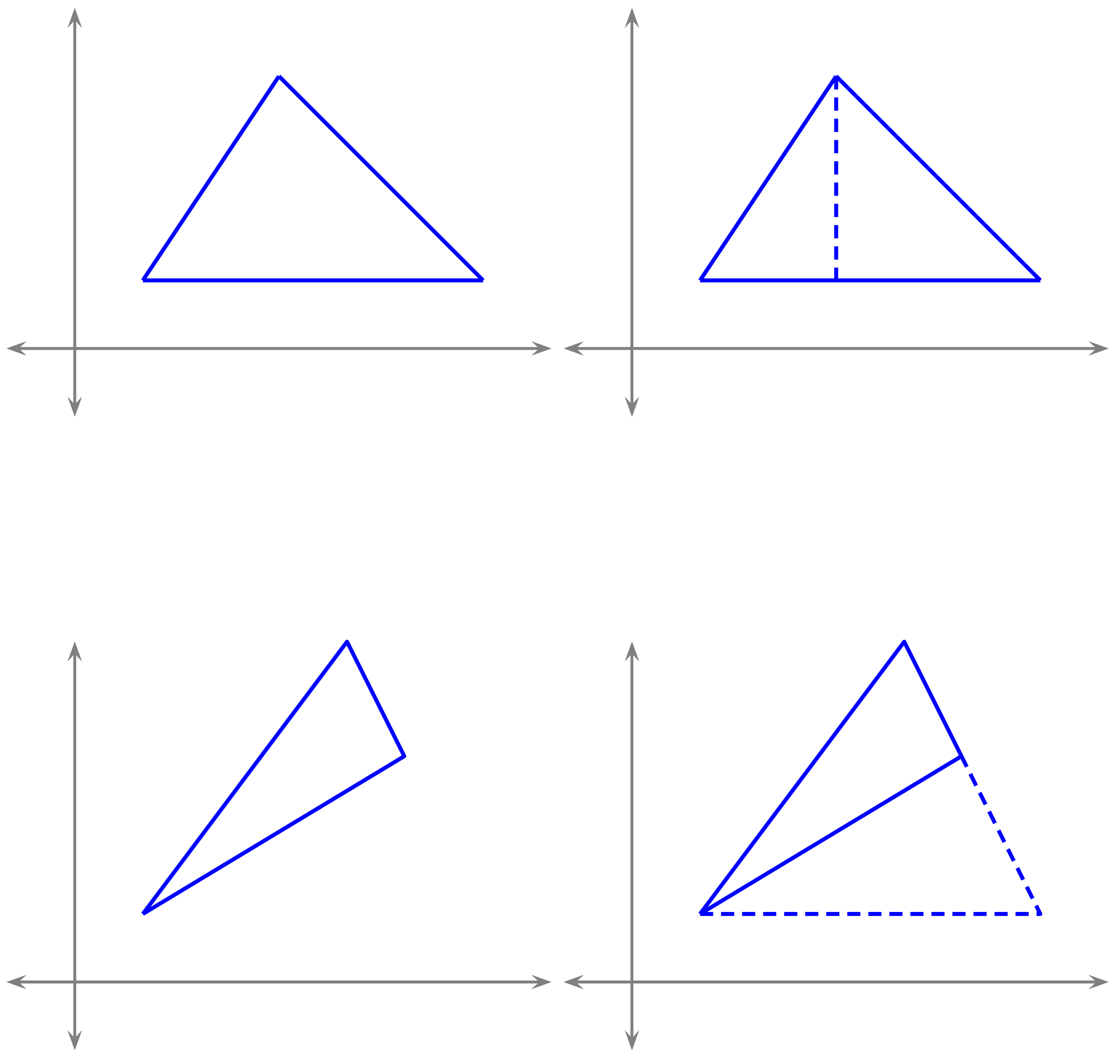
$$\int_{C_3} N\,dy = \int_d^c N\left(g^{-1}(y),y\right)\,dy = -\int_c^d N\left(g^{-1}(y),y\right)\,dy$$

   Putting these together we have

$$\oint_C M\,dx + N\,dy = \int_a^b \left(M(x,c) - M\left(x,g(x)\right)\right)\,dx + \int_c^d \left(N(b,y) - N\left(g^{-1}(y),y\right)\right)\,dy$$

   which is the RHS of (5).

Remark. Now to prove the theorem for arbitrary triangles, we first consider the following.



From this we could prove the theorem for arbitrary polygonal regions, etc.

Example 1. Consider the velocity vector field below.

$$\mathbf{F} = 3x\,\mathbf{j}$$

over the unit square R as shown in Figure 1. We imagine the field represents a thin fluid flowing over the xy-plane and the units of $\mathbf{F}$ are expressed in ft/sec.

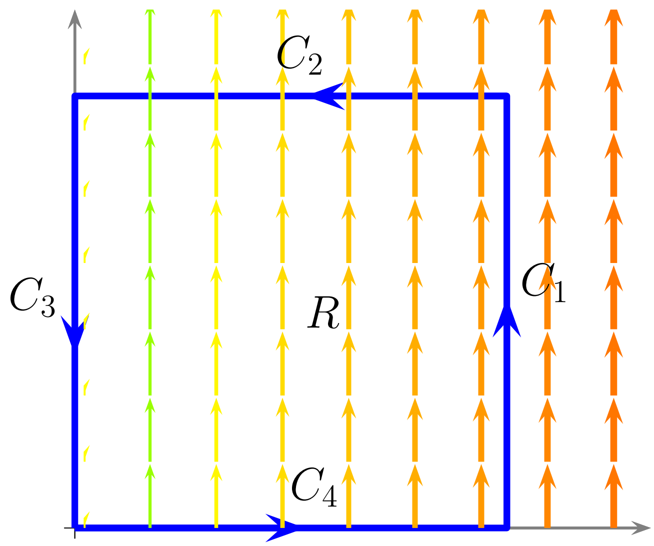


Figure 1: Velocity Field $3x\,\mathbf{j}$

What can you say about the circulation density (see Figure 1) at each point within the region R?

Now find the counterclockwise circulation for the velocity field $\mathbf{F}$ on ∂R, the boundary of R, in two different ways.

We first proceed directly, that is, we evaluate the line integral $\int_{\partial R} \mathbf{F} \cdot d\mathbf{r}$.

$$\int_{\partial R} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} \mathbf{F} \cdot d\mathbf{r}$$

Notice that the flow along each of the line segments C_2, C_3, C_4 is zero.

Now let

$$C_1\colon \quad \mathbf{r}_1(t) = \mathbf{i} + t\,\mathbf{j}, \quad 0 \le t \le 1$$

Then

$$\begin{aligned}\int_{\partial R} \mathbf{F}\cdot d\mathbf{r} &= \int_{C_1} \mathbf{F}(\mathbf{r}_1(t))\cdot d\mathbf{r}_1 \\ &= \int_0^1 3\times 1\,dt = 3\end{aligned}$$

Now find the circulation using Green's Theorem. We have

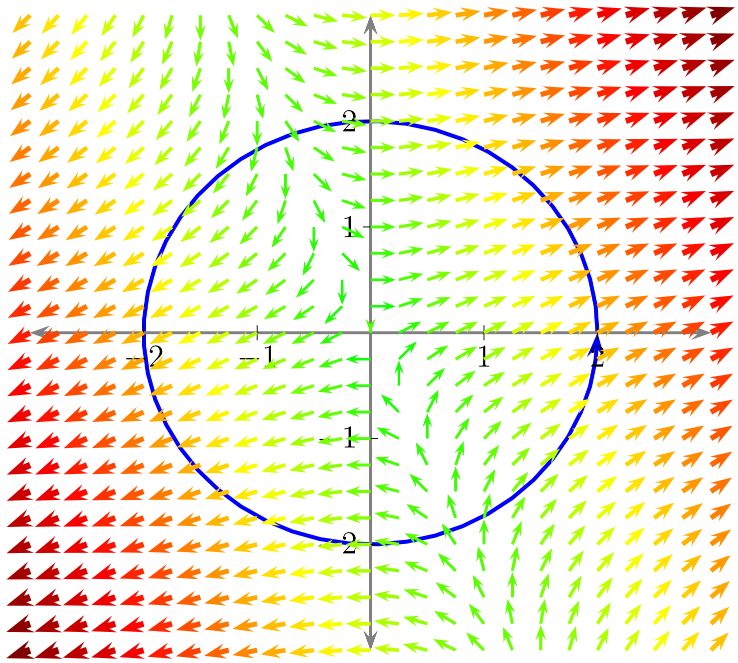
$$\begin{aligned}\int_{\partial R} \mathbf{F}\cdot d\mathbf{r} &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\,dx\,dy \\ &= \iint_R \left(\frac{\partial(3x)}{\partial x} - \frac{\partial(0)}{\partial y}\right)\,dx\,dy \\ &= 3\iint_R dx\,dy \\ &= 3\times \text{area of } R \\ &= 3\end{aligned}$$

as we saw above.

Example 2.

Let $f(x, y) = x^2 + xy$. Verify Green's Theorem for the gradient field $\nabla f = (2x + y)\,\mathbf{i} + x\,\mathbf{j}$ over the region R bounded by the circle

$$C\colon \mathbf{r}(t) = 2\,\cos t\,\mathbf{i} + 2\,\sin t\,\mathbf{j}, \quad 0 \le t \le 2\pi$$



So by Green's Theorem

$$\oint_C (2x + y)\,dx + x\,dy = \oint_C M\,dx + N\,dy$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\,dx\,dy$$

$$= \iint_R (1 - 1)\,dx\,dy$$

$$= 0$$

As expected since gradient vector fields are conservative.

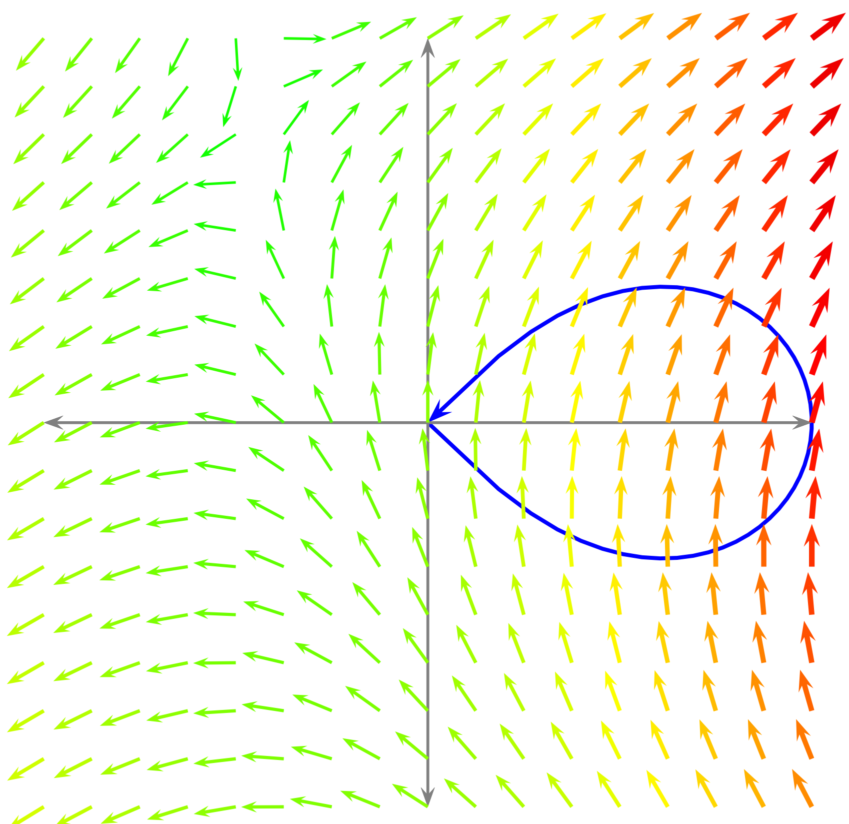
Example 3.

Use Green's Theorem to find the counterclockwise circulation for

$$\mathbf{F} = (x + e^x \sin y)\,\mathbf{i} + (x + e^x \cos y)\,\mathbf{j}$$

over the righthand loop of the lemniscate

$$C\colon r^2 = \cos 2\theta.$$



$$M = x + e^x \sin y, \quad N = x + e^x \cos y$$

$$\frac{\partial N}{\partial x} = 1 + e^x \cos y, \quad \frac{\partial M}{\partial y} = e^x \cos y$$

It follows that the circulation density is

$$\begin{aligned}(\nabla \times \mathbf{F}) \cdot \mathbf{k} &= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \\ &= 1 + e^x \cos y - e^x \cos y \\ &= 1\end{aligned}$$

So by Green's Theorem

$$\begin{aligned}\oint_C M\,dx + N\,dy &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \\ &= \iint_R 1\,dx\,dy \quad \text{(area of the lemniscate)} \\ &= \ldots \\ &= \frac{1}{2}\end{aligned}$$

Example 4.

Evaluate the circulation integral $\oint_C (3y\,dx + 2x\,dy)$ where C is boundary of the region

$$0 \le x \le \pi,\ 0 \le y \le \sin x$$

So let $\mathbf{F} = 3y\,\mathbf{i} + 2x\,\mathbf{j}$. See Figure 2.

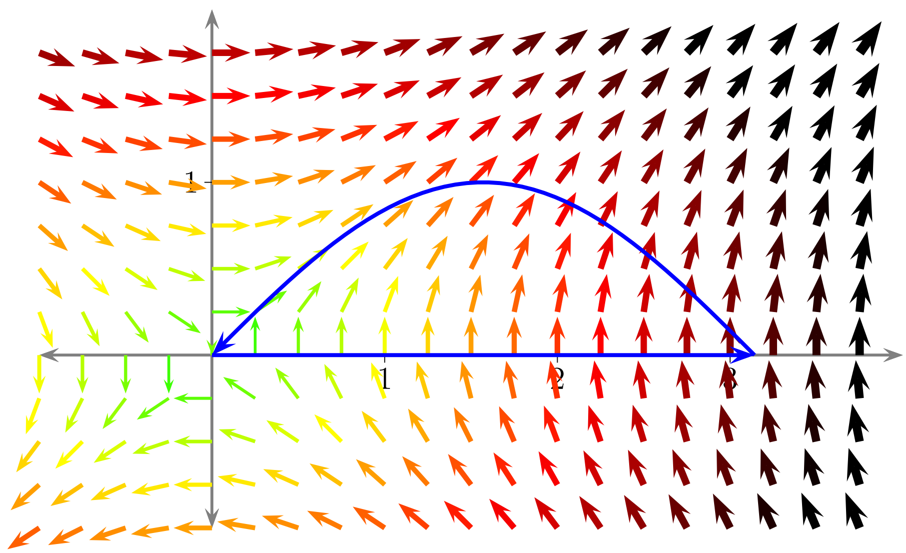


Figure 2: Vector Field $3y\,\mathbf{i} + 2x\,\mathbf{j}$

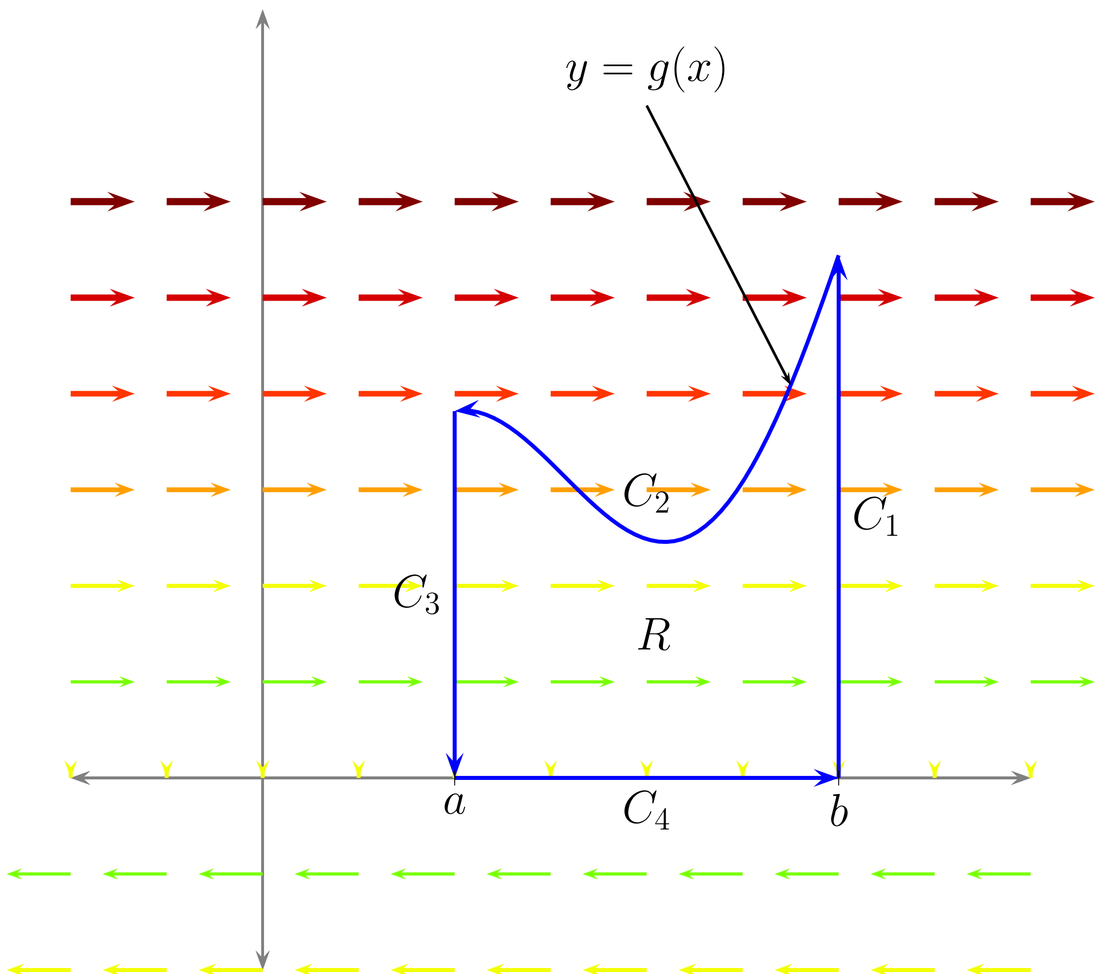
Now apply Green's Theorem.

$$\begin{aligned}
\oint_C 3y\,dx + 2x\,dy &= \oint_C M\,dx + N\,dy \\
&= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\,dx\,dy \\
&= \int_0^{\pi}\int_0^{\sin x} (2-3)\,dy\,dx \\
&= -\int_0^{\pi} \sin x\,dx \\
&= \cos x\,\Big|_0^{\pi} \\
&= -2
\end{aligned}$$

as we saw in section 16.3.

Example 5. Applying Green's Theorem - Special Results

a. Let $\mathbf{F} = y\,\mathbf{i}$. Let C be the boundary of the region R in the sketch below. Apply Green's Theorem to quickly find the circulation of $\mathbf{F}$ around C.



$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{T}\,ds &= \oint_C M\,dx + N\,dy \\ &= \oint_C y\,dx \\ &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\,dx\,dy \\ &= \int_a^b \int_0^{g(x)} (0-1)\,dy\,dx \\ &= -\int_a^b g(x)\,dx\end{aligned}$$

You can verify this result by actually computing the line integral using the parameterizations given below for C_1–C_4.

Notice that $C = C_1 \cup C_2 \cup C_3 \cup C_4$ where

$$C_1\colon \mathbf{r}_1(t) = b\,\mathbf{i} + g(b)t\,\mathbf{j}$$

$$C_2\colon \mathbf{r}_2(t) = (b-(b-a)\,t)\,\mathbf{i} + g(b-(b-a)\,t)\,\mathbf{j}$$

$$C_3\colon \mathbf{r}_3(t) = a\,\mathbf{i} + (1-t)g(a)\,\mathbf{j}$$

$$C_4\colon \mathbf{r}_4(t) = (a+(b-a)t)\,\mathbf{i},$$

Here each parametrization is given for $0 \le t \le 1$. We leave the remaining calculations as an exercise.

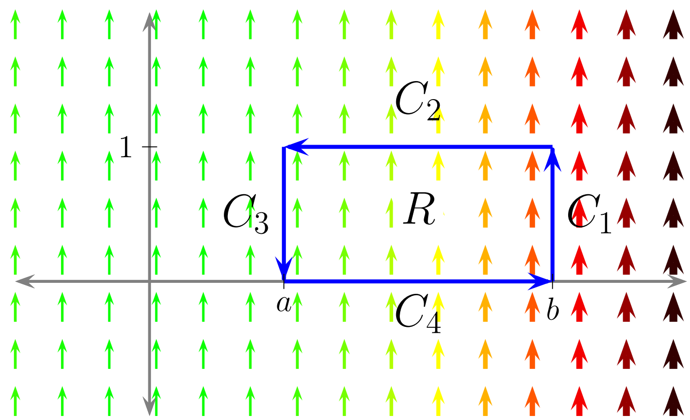


Figure 3: $\mathbf{F} = G(x)\,\mathbf{j}$

b. Let $\mathbf{F} = G(x)\,\mathbf{j}$ where G is a differentiable function of x. Let C be the boundary of a rectangle R shown in the sketch below. Apply Green's Theorem to obtain a well known result from an introductory calculus course.

The vector field is *constant* along vertical lines. In other words, if $c \in \mathbb{R}$,

$$\mathbf{F}(c, y) = G(c)\,\mathbf{j}, \quad \text{for all } y \in \mathbb{R}$$

First we compute the circulation of $\mathbf{F} = G(x)\,\mathbf{j}$ around the closed curve C directly.

Notice that $\mathbf{F} \cdot \mathbf{T} = 0$ for the line segments C_2 and C_4. Also, $\Delta y_{C_1} = \Delta y_{C_3} = 1$. It follows that

$$\begin{aligned} \text{circulation} &= (\mathbf{F}(b, y) \cdot \mathbf{j})\ \Delta y + (\mathbf{F}(a, y) \cdot (-\mathbf{j}))\ \Delta y \\ &= G(b)\,\Delta y - G(a)\,\Delta y \\ &= G(b) - G(a) \end{aligned}$$

Now, $\dfrac{\partial N}{\partial x} = \dfrac{\partial G(x)}{\partial x} = G'(x) = g(x)$ and $\dfrac{\partial M}{\partial y} = \dfrac{\partial (0)}{\partial y} = 0$.

So by Green's Theorem

$$\begin{aligned} G(b) - G(a) &= \int_C \mathbf{F} \cdot \mathbf{T}\, ds \\ &= \oint_C M\, dx + N\, dy \\ &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy \\ &= \int_a^b \int_0^1 (G'(x) - 0)\, dy\, dx \\ &= \int_a^b G'(x)\, dx \\ &= \int_a^b g(x)\, dx \end{aligned}$$

So the Fundamental Theorem of Calculus is a special case of (the tangential form) Green's Theorem.

In this case the definite integral $\int_a^b g(x)\, dx$ can be viewed as the *circulation* of $\mathbf{F} = G(x)\, \mathbf{j}$ around the closed curve C (see Fig. 3) where $G(x)$ is any antiderivative of $g(x)$.

Applications

Green's Theorem can be used to derive several useful formulas for area.

Let C be a simple closed curve in the plane enclosing a region R. Then

$$\begin{aligned}
\text{Area of } R &= \iint_R dx\,dy \\
&= \frac{1}{2}\iint_R (1-(-1))\;dx\,dy \\
&= \frac{1}{2}\iint_R \left(\frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y}\right)\;dx\,dy \\
&= \frac{1}{2}\oint_C -y\,dx + x\,dy \qquad (6)
\end{aligned}$$

The above formula can be used to explain how a planimeter works (see the brief discussion and references on page 1111 of the text).

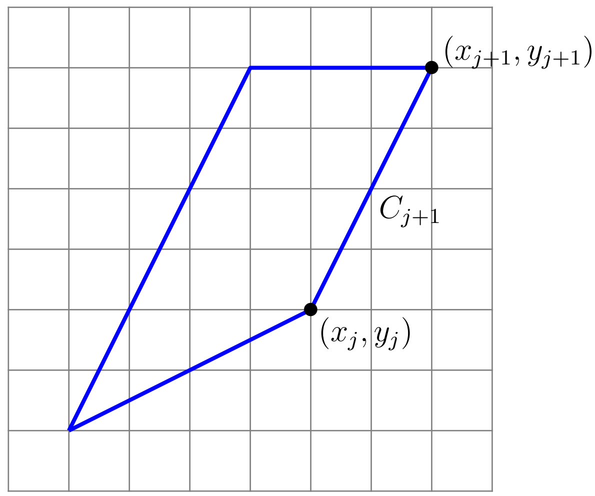


Figure 4: General Polygonal Region

Example 6. Let R be a polygon with vertices $(x_0, y_0), (x_1, y_1), \ldots (x_n, y_n)$. A typical edge is shown in Figure 4. Use (6) to derive the formula

$$\text{(7)} \qquad \text{Area of } R = \frac{1}{2} \sum_{j=0}^{n-1} x_j y_{j+1} - x_{j+1} y_j$$

So by (6), the area of the polygon is given by

$$\begin{aligned} \text{Area} &= \frac{1}{2} \oint_C x\,dy - y\,dx \\ &= \frac{1}{2} \oint_{\cup_{j=1}^n C_j} x\,dy - y\,dx \\ &= \frac{1}{2} \int_{C_1} x\,dy - y\,dx + \frac{1}{2} \int_{C_1} x\,dy - y\,dx + \\ &\qquad \cdots + \frac{1}{2} \int_{C_n} x\,dy - y\,dx \end{aligned}$$

We compute the flow integral along an arbitrary side. So let

$$C_{j+1} \colon \mathbf{r}(t) = (x_j(1-t) + x_{j+1}t)\ \mathbf{i} + (y_j(1-t) + y_{j+1}t)\ \mathbf{j}, \quad 0 \le t \le 1$$

Then

$$x = x_j(1-t) + x_{j+1}t, \quad dy = (y_{j+1} - y_j)\ dt$$
$$y = y_j(1-t) + y_{j+1}t, \quad dx = (x_{j+1} - x_j)\ dt$$

It follows that

$$\begin{aligned}\frac{1}{2}\oint_{C_{j+1}} x\,dy - y\,dx &= \frac{1}{2}\int_0^1 \left[(x_j(1-t) + x_{j+1}t)\,(y_{j+1} - y_j) - (y_j(1-t) + y_{j+1}t)\ (\right. \\ &= \frac{1}{2}\int_0^1 (x_j y_{j+1} - x_{j+1} y_j)\ dt \\ &= \frac{1}{2} x_j y_{j+1} - \frac{1}{2} x_{j+1} y_j\end{aligned}$$

Thus

$$\begin{aligned}\text{Area} &= \frac{1}{2}\int_{C_1} x\,dy - y\,dx + \frac{1}{2}\int_{C_1} x\,dy - y\,dx + \\ &\qquad \cdots + \frac{1}{2}\int_{C_n} x\,dy - y\,dx \\ &= \frac{1}{2}\sum_{j=0}^{n-1}\int_{C_{j+1}} x\,dy - y\,dx \\ &= \frac{1}{2}\sum_{j=0}^{n-1} x_j y_{j+1} - x_{j+1} y_j\end{aligned}$$

as desired.

As another application of (6), we derive the well-known formula for the area of an ellipse. Recall that the general equation of an ellipse centered at the origin is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

for some $a, b > 0$. It is easy to see that this ellipse can be parameterized by

$$C\colon\ \mathbf{r}(t) = a\cos t\,\mathbf{i} + b\sin t\,\mathbf{j}, \quad 0 \le t \le 2\pi$$

Now let R be the region enclosed by the ellipse. Then by (6)

$$\begin{aligned}\text{Area of ellipse} &= \frac{1}{2}\oint_C x\,dy - y\,dx \\ &= \frac{1}{2}\int_0^{2\pi} \left((a\cos t)(b\cos t) - (b\sin t)(-a\sin t)\right)\,dt \\ &= \frac{ab}{2}\int_0^{2\pi} dt = \pi ab\end{aligned}$$

How would you calculate the area using techniques from first semester calculus?

Example 7.

Evaluate the integral $\oint_C 4xy\,dx + x\,dy$. Here C is defined by the vector equation

(8) $$C: \quad \mathbf{r}(t) = -2\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j}, \quad 0 \le t \le 2\pi$$

Let $\mathbf{F} = 4xy\,\mathbf{i} + x\,\mathbf{j}$. Then the given integral is a circulation integral.

So let R be the interior of the ellipse defined in (8). Then by (the tangential form of) Green's Theorem

$$\oint_C 4xy\,dx + x\,dy = \iint_R \left(\frac{\partial(x)}{\partial x} - \frac{\partial(4xy)}{\partial y}\right)\,dx\,dy$$
$$= \iint_R (1-4x)\,dx\,dy$$

Notice that the ellipse has the rectangular equation

(9) $$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

Thus

$$\iint_R (1-4x)\,dx\,dy = \int_{-2}^{2}(1-4x)\int_{-\sqrt{36-9x^2}/2}^{\sqrt{36-9x^2}/2} dy\,dx$$
$$= \int_{-2}^{2}(1-4x)\sqrt{36-9x^2}\,dx$$

We try the trig substitution $x = 2\sin\theta$. Then

$$\begin{aligned}
\int_{-2}^{2}(1-4x)\sqrt{36-9x^2}\,dx &= 12\int_{-\pi/2}^{\pi/2}(1-8\sin\theta)\cos^2\theta\,d\theta \\
&= 6\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta - 96\int_{-\pi/2}^{\pi/2}\sin\theta\ \cos^2\theta\,d\theta \\
&= 6\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta - 0 \\
&= \vdots \\
&= 6\pi
\end{aligned}$$

We leave it as an exercise to evaluate the line integral directly.

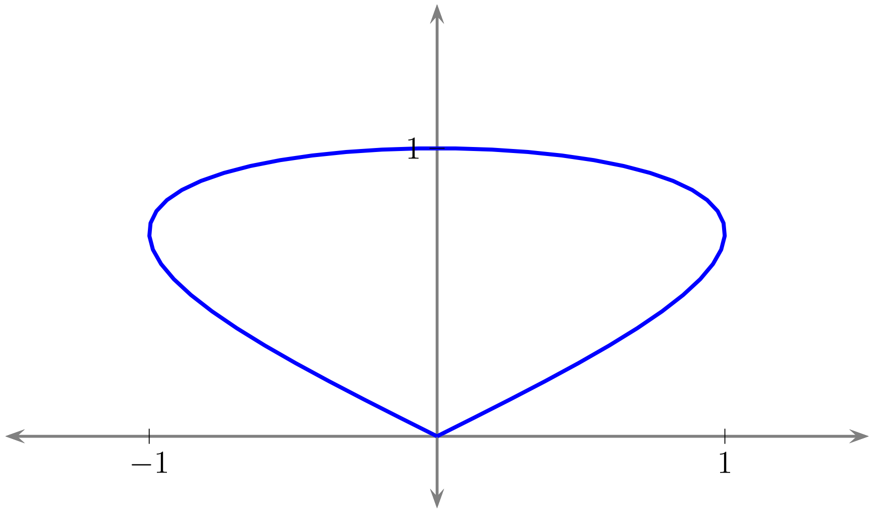


Figure 5: Parametric Curve: $\mathbf{r} = \sin 2t\ \mathbf{i} + \sin t\,\mathbf{j}$

Example 8. Find the area of the region R whose boundary is given by the vector equation $\mathbf{r}(t) = \sin 2t\ \mathbf{i} + \sin t\ \mathbf{j},\ 0 \leq t \leq \pi$.

So by (6), this is

$$\begin{aligned}\text{Area} &= \frac{1}{2}\oint_C -y\,dx + x\,dy \\ &= \frac{1}{2}\int_0^{\pi} -2\sin t\cos 2t + \sin 2t\cos t\,dt \\ &= \int_0^{\pi} -\sin t(2\cos^2 t - 1) + \sin t\cos^2 t\,dt \\ &= \int_1^{-1} u^2 - 1\,du \\ &= \frac{4}{3}\end{aligned}$$

The solution above is nice and short but it sure seems like overkill (using Green's Theorem) for a simple area calculation. Can we use one of the definitions from sections 15.3 or 15.4 to compute the area of R?

Example 9. Redo the previous example by rewriting the given vector equation in polar form.

Let's rewrite the parametric equations

(10) $$x = \sin 2t = 2\sin t\ \cos t \text{ and } y = \sin t$$

in polar form and exploit the formula $A = \iint_R r\,dr\,d\theta$. Now

$$\begin{aligned} r^2 = x^2 + y^2 &= \sin^2 t(4\cos^2 t + 1) \\ &= (1 - \cos^2 t)(4\cos^2 t + 1) \end{aligned}$$

and

$$\tan\theta = \frac{y}{x} = \frac{1}{2\cos t}$$

or

$$\cos t = \frac{\cot\theta}{2}$$

Let $\theta_0 = \arctan(1/2)$. It follows that

$$r(\theta) = \sqrt{\left(1 - \frac{\cot^2\theta}{4}\right)(1 + \cot^2\theta)},$$

$$\theta_0 \le \theta \le \pi - \theta_0$$

Now

$$\begin{aligned} \text{Area} = \iint_R r\,dr\,d\theta &= \int_{\theta_0}^{\pi-\theta_0} \int_0^{r(\theta)} r\,dr\,d\theta \\ &= 2\int_{\theta_0}^{\pi/2} \int_0^{r(\theta)} r\,dr\,d\theta \end{aligned}$$

$$
\begin{aligned}
\text{Area} &= \int_{\theta_0}^{\pi/2} r^2(\theta)\,d\theta \\
&= \int_{\theta_0}^{\pi/2} \left(1 - \frac{\cot^2\theta}{4}\right)(1+\cot^2\theta)\,d\theta \\
&= \int_{\theta_0}^{\pi/2} 1 + \frac{3\cot^2\theta}{4} - \frac{\cot^4\theta}{4}\,d\theta \\
&= \vdots \\
&= \frac{4}{3}
\end{aligned}
$$

As we saw above. Here we have relied on the standard trig reductions formulas for the cotangent function. For example,

$$
\begin{aligned}
\int \cot^4\theta\,d\theta &= \int \cot^2\theta(\csc^2\theta - 1)\,d\theta \\
&= \int \cot^2\theta\csc^2\theta\,d\theta - \int \cot^2\theta\,d\theta \\
&= \int \cot^2\theta\csc^2\theta\,d\theta - \int (\csc^2\theta - 1)\,d\theta \\
&= \frac{-\cot^3\theta}{3} + \cot\theta + \theta + C
\end{aligned}
$$

That’s a lot of work for a simple area calculation. Perhaps working in rectangular coordinates would be easier.

Example 10. Redo Example 8 by working in rectangular coordinates.

This turns out to be easier than working in polar coordinates. Notice that $x = f(y)$ since by (10) we have

$$x = 2\sin t\ \cos t = 2\sin t\sqrt{1-\sin^2 t}$$
$$= 2y\sqrt{1-y^2}$$

Now the rest is easy. We have

$$\text{Area} = 2\int_0^1 \int_0^{2y\sqrt{1-y^2}} dx\, dy$$
$$= 2\int_0^1 2y\sqrt{1-y^2}\, dy$$
$$= -2\int_1^0 \sqrt{u}\, du$$
$$= \frac{4}{3}u^{3/2}\Big|_0^1$$
$$= \frac{4}{3}$$

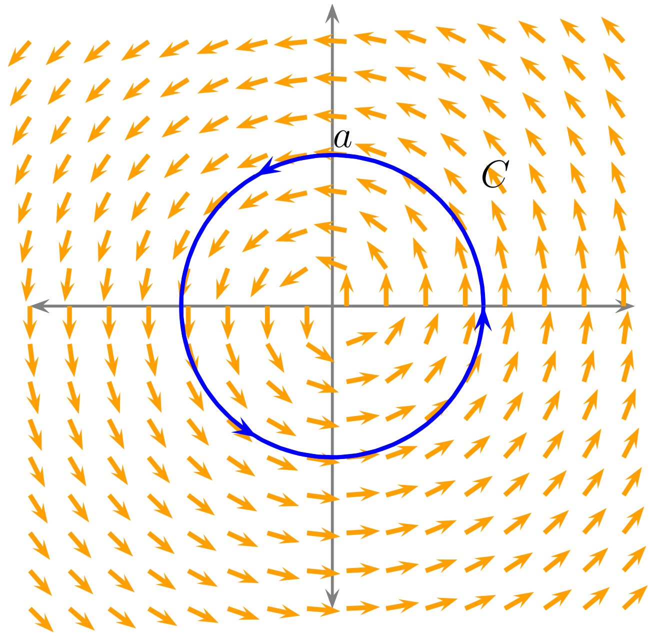


Figure 6: A Spin Field

Example 11. Let $\mathbf{F} = \dfrac{-y}{x^2+y^2}\,\mathbf{i} + \dfrac{x}{x^2+y^2}\,\mathbf{j}$ and let C be a positively oriented circle of radius $a > 0$ centered at the origin. Evaluate the circulation integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

As usual C can be parameterized by the vector equation $\mathbf{r}(t) = a\cos t\ \mathbf{i} + a\sin t\ \mathbf{j},\ 0 \le t \le 2\pi$. It follows that $d\mathbf{r}/dt = \langle -a\sin t, a\cos t\rangle$ and

$$\mathbf{F}(\mathbf{r}(t)) = \frac{-\sin t}{a}\,\mathbf{i} + \frac{\cos t}{a}\,\mathbf{j}$$

so that

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \sin^2 t + \cos^2 t\, dt \\ &= \int_0^{2\pi} dt = 2\pi\end{aligned}$$

Compare with Example 7 from 16.2.

Now let's try using Green's Theorem. Let D be the disk of radius $a > 0$ centered at the origin and let

$$M = \frac{-y}{x^2 + y^2} \quad \text{and} \quad N = \frac{x}{x^2 + y^2}$$

Then according to Green's Theorem

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy \\
&= \iint_D \left(\frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2} \right) dx\, dy \\
&= \iint_D 0\, dx\, dy \\
&= 0 \text{ ???}
\end{aligned}$$

What is going on?

We will have more to say about this example next time.