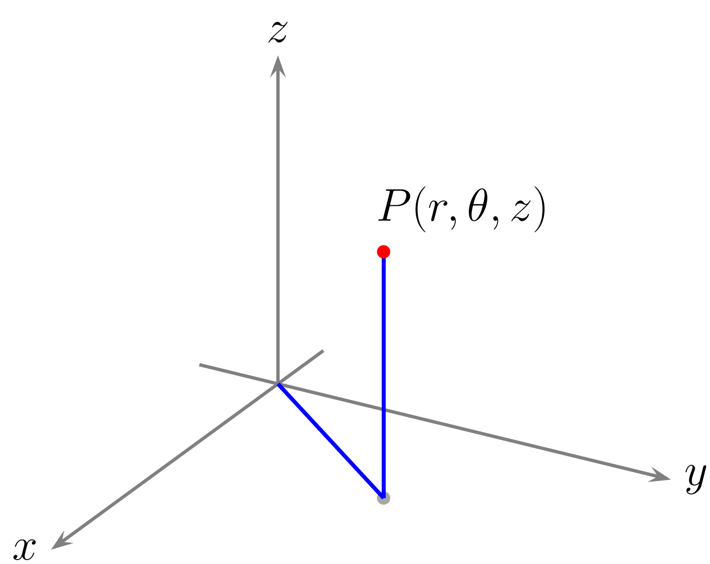
15.8 Triple Integrals in Cylindrical Coordinates

Integration in Cylindrical Coordinates

Definition. **Cylindrical coordinates** represent a point P in space by the ordered triple (r, θ, z) where

1. r and θ are the polar coordinates for the vertical projection of P onto the xy-plane.
2. z is the rectangular vertical coordinate of P.



The following equations relate rectangular coordinates (x, y, z) to cylindrical coordinates (r, θ, z).

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

$$\big(\text{Also, } r^2 = x^2 + y^2 \;\text{ and }\; \tan\theta = y/x\big)$$

Remark. One must exercise care when using the second set of equations.

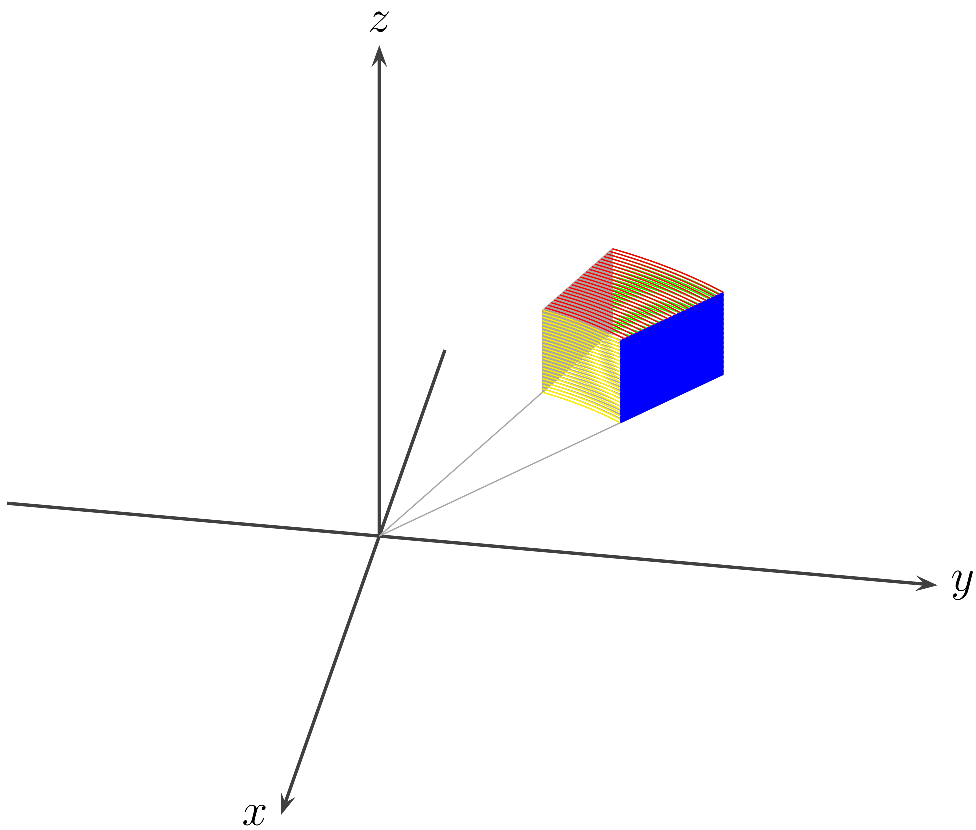
Example 1. Constant-Coordinate Equations

Describe the objects generated by the constant equations:

$$r = r_0$$

$$\theta = \theta_0$$

$$z = z_0$$



Suppose that $f(r, \theta, z)$ is defined on a closed bounded region D in space. Can we define the integral of f over D? Proceeding in the usual way (that is, partitioning the region D, etc.), we obtain the following (Riemann) sum

$$S_n = \sum_{k=1}^{n} f\left(r_k, \theta_k, z_k\right) \triangle V_k$$

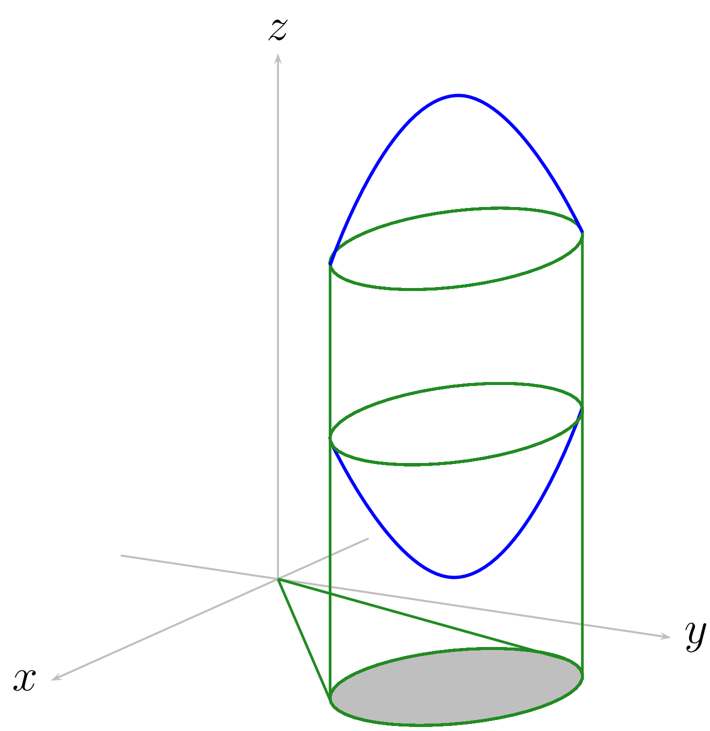
where $\triangle V_k = \triangle z_k \, r_k \, \triangle r_k \, \triangle \theta_k$.

Now we take the limit of the above expression as $\|P\| \to 0$, where $\|P\|$ is the norm of the partition P. If the limit exists we say that f **is integrable over** D and write

$$\begin{aligned}\lim_{n\to\infty} S_n &= \iiint_D f\,dV \\ &= \iiint_D f(r,\theta,z)\,dz\,r\,dr\,d\theta\end{aligned}$$

It turns out that if f is continuous over the closed bounded region D then f is integrable (as long as D is “reasonable”). (See also the remarks following Example 2 below.)

Finding the limits of integration in cylindrical coordinates.

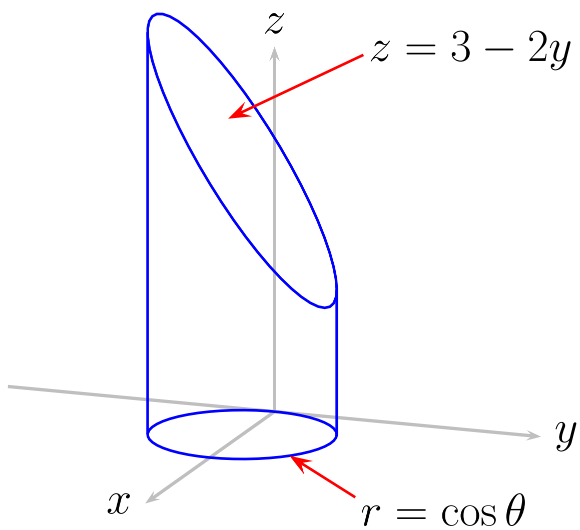


If $f(r, \theta, z)$ is continuous over a region $D \in \mathbb{R}^3$ then

$$\iiint_D f\,dV = \iiint_D f(r,\theta,z)\,dz\,r\,dr\,d\theta$$
$$= \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r,\theta,z)\,dz\,r\,dr\,d\theta$$

Example 2. Integrating in Cylindrical Coordinates

Let D be the solid right cylinder whose base is the region inside the circle (in the xy-plane) $r = \cos\theta$ and whose top lies in the plane $z = 3 - 2y$ (see sketch).



a. Set up the triple integral in cylindrical coordinates that gives the volume of D.

$$\iiint_D dV = \int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}\int_0^{3-2y} r\,dz\,dr\,d\theta$$

Of course, $3 - 2y = 3 - 2r\,\sin\theta$.

b. Find the volume of D by evaluating the iterated integral from part (a).

$$\iiint_D dV = \int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}\int_0^{3-2y} r\,dz\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta} r(3-2y)\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta} \left(3r - 2r^2\sin\theta\right)\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{3r^2}{2} - \frac{2r^3\,\sin\theta}{3}\right)\Big|_0^{\cos\theta}\,d\theta$$

$$= \frac{1}{6}\int_{-\pi/2}^{\pi/2} \left(9\cos^2\theta - 4\cos^3\theta\,\sin\theta\right)\,d\theta$$

$$= \frac{9}{6}\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta - \frac{4}{6}\int_{-\pi/2}^{\pi/2}\cos^3\theta\,\sin\theta\,d\theta$$

$$= \frac{3}{4}\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta + \frac{2}{3}\int_0^0 u^3\,du$$

The second integral above is obviously 0. Thus

$$\iiint_D dV = \frac{3}{4}\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta + 0$$

$$= \frac{3}{4}\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_{-\pi/2}^{\pi/2} = \frac{3\pi}{4}$$

Remark. It is easy to check that

$$\int_0^{\pi}\int_0^{\cos\theta}\int_0^{3-2y} r\,dz\,dr\,d\theta = \frac{3\pi}{4}$$

However, one must proceed with caution as the following example illustrates. We leave the evaluation of the following integral as an exercise.

$$\int_0^1\int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} \tan^{-1}(y/x)\,dy\,dx = 0 \tag{1}$$

What happens when we convert (1) to the equivalent integral in polar coordinates.

$$\begin{aligned}\int_0^1\int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} \tan^{-1}(y/x)\,dy\,dx &= \int_{\alpha}^{\beta}\int_0^{\cos\theta} \theta\,r\,dr\,d\theta \\ &= \frac{1}{2}\int_{\alpha}^{\beta} \theta\cos^2\theta\,d\theta \qquad (2)\end{aligned}$$

We now have two obvious choices for α and β.

If we let $\alpha = -\pi/2$ and $\beta = \pi/2$, then the integral in (2) evaluates to 0 since integrand $\theta\cos^2\theta$ is odd.

On the other hand, it is easy to see that

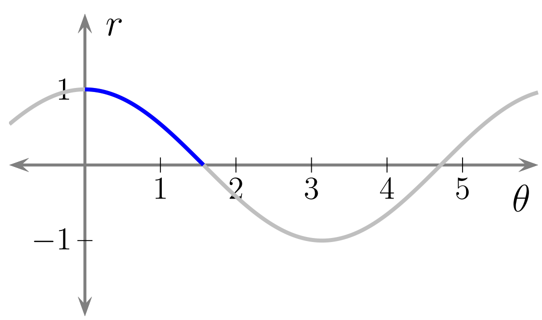
$$\frac{1}{2}\int_0^{\pi} \theta\cos^2\theta\,d\theta > 0$$

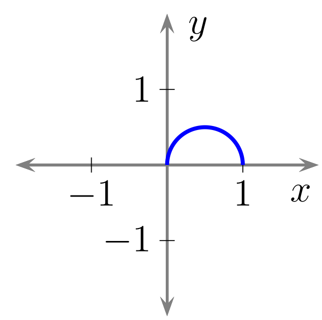
since $\theta\cos^2\theta > 0$ for $0 < \theta < \pi/2$ and $\pi/2 < \theta < \pi$. This in contrary to the result above. It is beyond the scope of the course to go into too much detail about this issue. So we conclude with a simple warning to use caution when evaluating similar integrals.

Example 3. Explain why the limits of integration of the outside integral in the previous example must be $\theta = 0$ to $\theta = \pi$. Or more precisely, why they should be $\theta = -\pi/2$ to $\theta = \pi/2$.

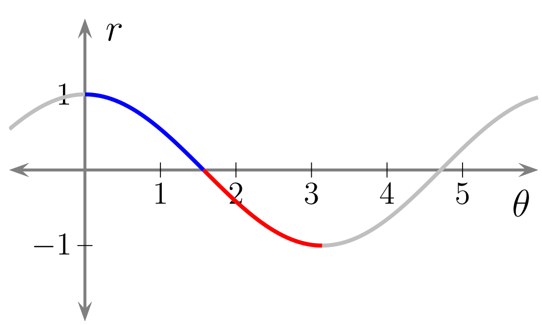
To see this we sketch the polar equation $r = \cos\theta$ by "plotting points". It's a bit easier to also sketch the graph of $r = \cos\theta$ in the $r\theta$-coordinate system instead of setting up a table of inputs, θ, and outputs, $r = f(\theta)$.

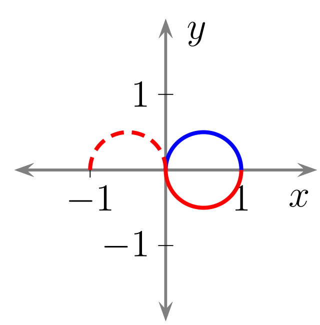
First try $0 \leq \theta \leq \dfrac{\pi}{2}$.





Now sketch the portion for $\dfrac{\theta}{2} \leq \theta \leq \pi$.





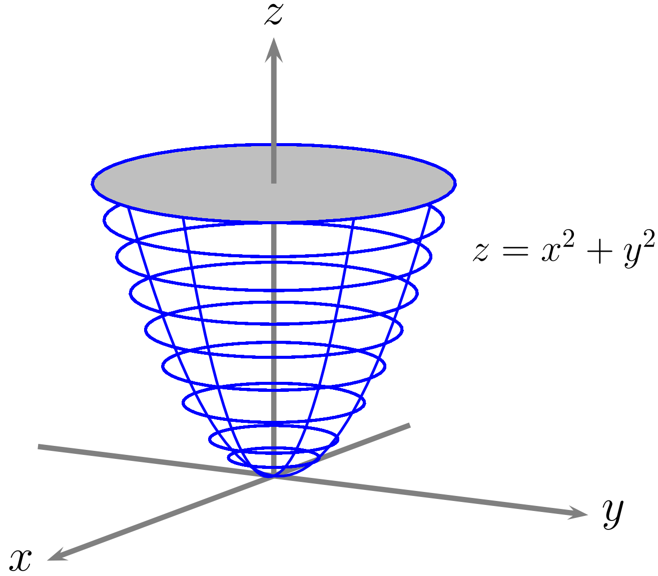
The dashed red curve in the sketch above is for the polar equation

$r = |\cos\theta|,\ \pi/2 \le \theta \le \pi$. Of course, we already knew that the graph of the given polar equation was a circle of radius $1/2$ centered at $(1/2, 0)$ since $r = \cos\theta \Longrightarrow r^2 = r\cos\theta$. Converting to rectangular coordinates we obtain

$$x^2 + y^2 = x \quad \Longrightarrow \quad (x - 1/2)^2 + y^2 = 1/4$$

However, we were unsure which values of θ were necessary to generate a complete circle. It follows just as easily that we could also "parameterize" the circle using $-\pi/2 \le \theta \le \pi/2$.

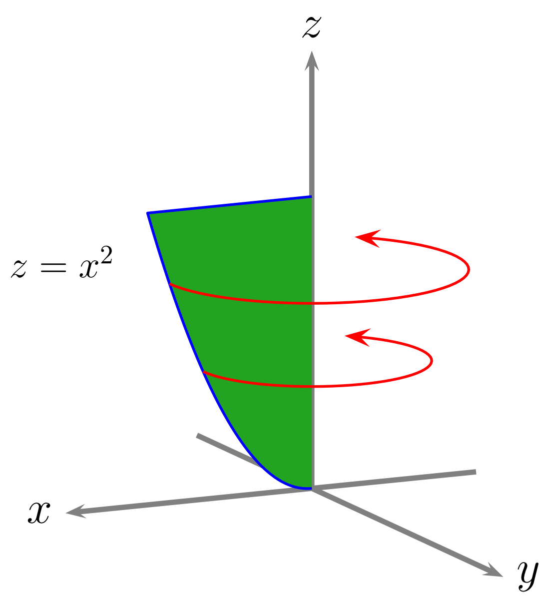
Example 4. Find the volume of the solid that lies between the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.



It follows that the volume is given by

$$\begin{aligned}
\text{V} &= \int_0^{2\pi} \int_0^2 \int_{x^2+y^2}^4 dz\, r dr\, d\theta \\
&= 2\pi \int_0^2 \int_{x^2+y^2}^4 dz\, r dr \\
&= 2\pi \int_0^2 (4 - x^2 - y^2)\, r dr \\
&= 2\pi \int_0^2 (4 - r^2) r dr \\
&= \vdots \\
&= 8\pi
\end{aligned}$$

Notice also that this solid can be recognized as a solid of revolution. In other words, we can use techniques from Calculus II to compute the volume.



Now the cross sections perpendicular to the z-axis are disks of radius $\sqrt{z}$. It follows that the cross-sectional area is given by the formula $A(z) = \pi\left(\sqrt{z}\right)^2 = \pi z$ and hence the volume of revolution is

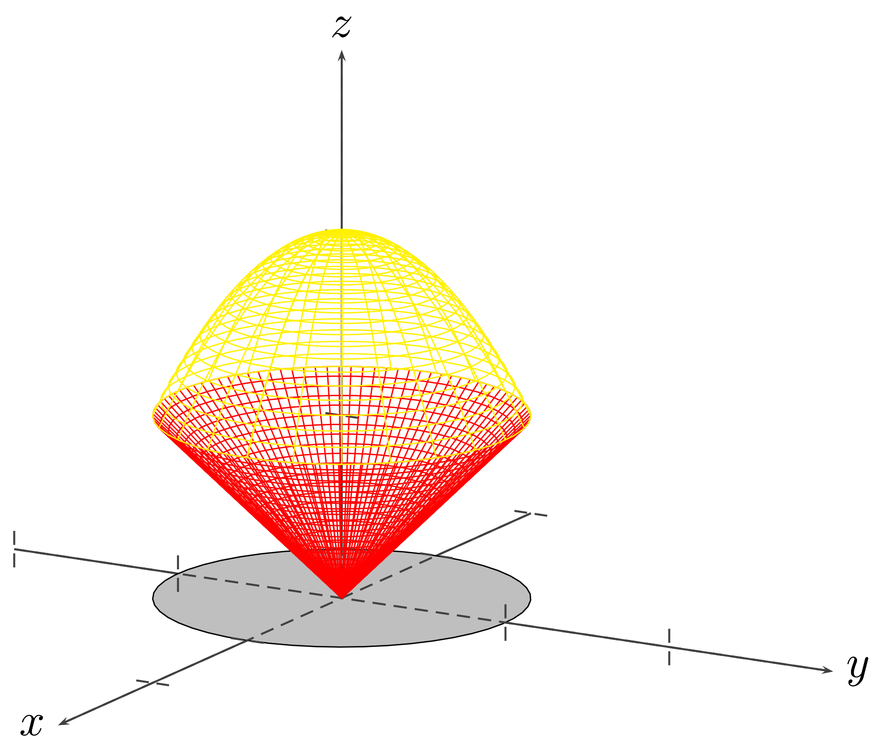
$$\begin{aligned} V &= \int_0^4 A(z)\,dz \\ &= \pi \int_0^4 z\,dz \\ &= \frac{\pi}{2} z^2 \Big|_0^4 \\ &= 8\pi \end{aligned}$$

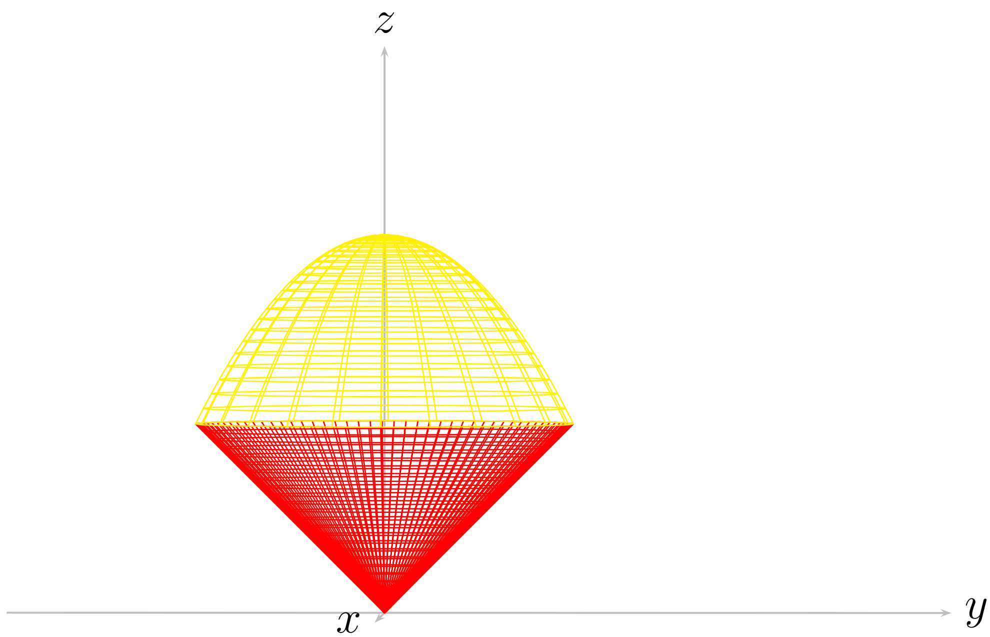
as we saw above.

Example 5. Ice-Cream Cone

Let D be the region (*an ice-cream cone*) bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$.

Set up the triple integral using cylindrical coordinates that give the volume using each of the following orders of integration.





Notice that the surfaces intersect at $z = 1$ and that the projection onto the xy-plane is the unit disk.

$$
\begin{aligned}
V &= \iiint_D r\,dz\,dr\,d\theta \\
&= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=r}^{z=2-r^2} r\,dz\,dr\,d\theta \\
&= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} r\left(2-r^2-r\right)\,dr\,d\theta \\
&= \int_{\theta=0}^{\theta=2\pi} \left(r^2 - \frac{r^4}{4} - \frac{r^3}{3}\right)\Bigg|_{r=0}^{r=1} d\theta \\
&= \int_{\theta=0}^{\theta=2\pi} \left(\frac{5}{12}\right)\,d\theta \\
&= \frac{5\pi}{6}
\end{aligned}
$$

Example 6. Switching the Order of Integration (Again)

Redo the last example by changing the order of integration.

a. First try $dr\,dz\,d\theta$

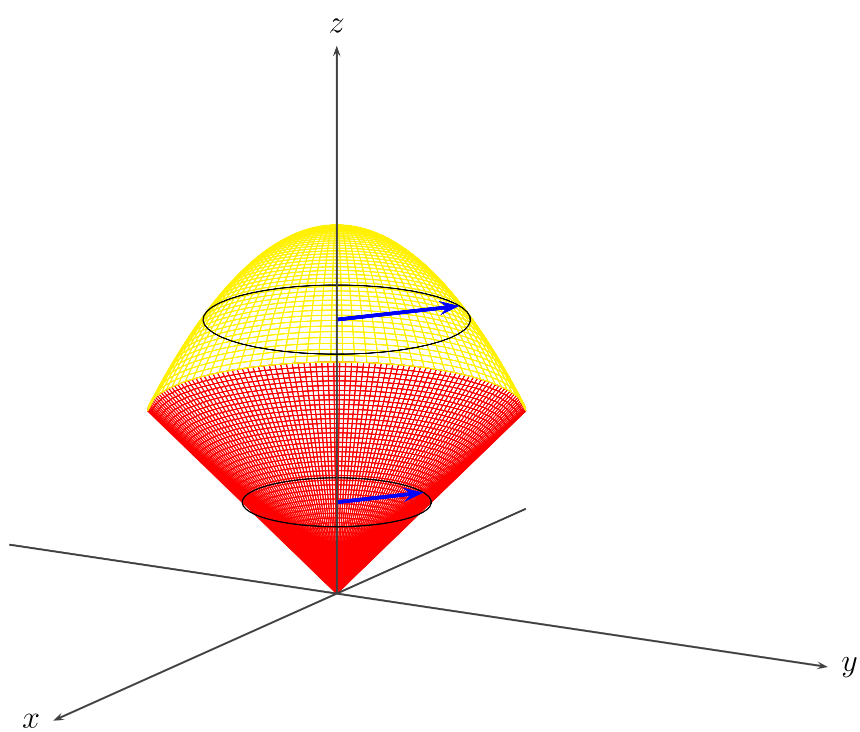


Figure 1: Cutaway View

Notice that if we first integrate with respect to r, we see that $0 \le r \le z$ for $0 \le z \le 1$ and $0 \le r \le \sqrt{2-z}$ if $1 \le z \le 2$ (see figure 1).

$$
\begin{aligned}
V &= \iiint_D r\,dr\,dz\,d\theta \\
&= \iiint_{\text{Cone}} r\,dr\,dz\,d\theta + \iiint_{\text{Cream}} r\,dr\,dz\,d\theta \\
&= \int_{\theta=0}^{\theta=2\pi}\int_{z=0}^{z=1}\int_{r=0}^{r=z} r\,dr\,dz\,d\theta \\
&\qquad + \int_{\theta=0}^{\theta=2\pi}\int_{z=1}^{z=2}\int_{r=0}^{r=\sqrt{2-z}} r\,dr\,dz\,d\theta \\
&= 2\pi\int_{z=0}^{z=1}\int_{r=0}^{r=z} r\,dr\,dz + 2\pi\int_{z=1}^{z=2}\int_{r=0}^{r=\sqrt{2-z}} r\,dr\,dz \\
&= 2\pi\int_0^1 \frac{z^2}{2}\,dz + 2\pi\int_1^2 \frac{2-z}{2}\,dz \\
&= 2\pi\left(\frac{z^3}{6}\,\Big|_0^1 + \frac{4z-z^2}{4}\,\Big|_1^2\right) \\
&= 2\pi\left(\frac{1}{6}+\frac{1}{4}\right) \\
&= \frac{5\pi}{6}
\end{aligned}
$$

b. $d\theta\,dz\,dr$

$$
\begin{aligned}
V &= \iiint_D r \, d\theta \, dz \, dr \\
&= \int_{r=0}^{r=1} \int_{z=r}^{z=2-r^2} \int_{\theta=0}^{\theta=2\pi} r \, d\theta \, dz \, dr \\
&= \dots \\
&= \frac{5\pi}{6}
\end{aligned}
$$

Example 7. Let $a > 0$ and let D be the solid cut by the cylinder $r = a\cos\theta$ and bounded above and below by a sphere of radius a centered at the origin. Express the volume of D as a triple integral in cylindrical coordinates and evaluate.

We have

$$
\begin{aligned}
V &= \int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta \\
&= 2\int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta \\
&= 2\int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta \\
&= \int_{-\pi/2}^{\pi/2}\int_{a^2\sin^2\theta}^{a^2} \sqrt{u}\,du\,d\theta, \qquad (u = a^2 - r^2, \text{ etc.}) \\
&= \frac{2}{3}\int_{-\pi/2}^{\pi/2} u^{3/2}\,\Big|_{a^2\sin^2\theta}^{a^2}\,d\theta \\
&= \frac{2a^3}{3}\int_{-\pi/2}^{\pi/2}\left(1 - |\sin\theta|^3\right)\,d\theta \\
&= \frac{2\pi a^3}{3} - \frac{4a^3}{3}\int_0^{\pi/2}\sin^3\theta\,d\theta \\
&\;\;\vdots \\
&= \frac{2a^3}{3}\left(\pi - \frac{4}{3}\right)
\end{aligned}
$$