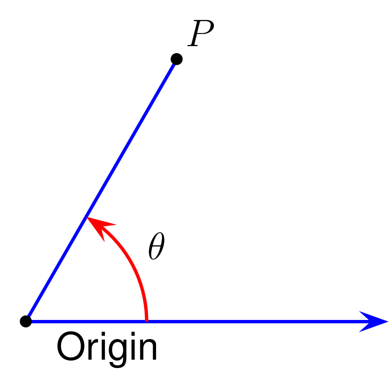
Double Integrals in Polar Coordinates

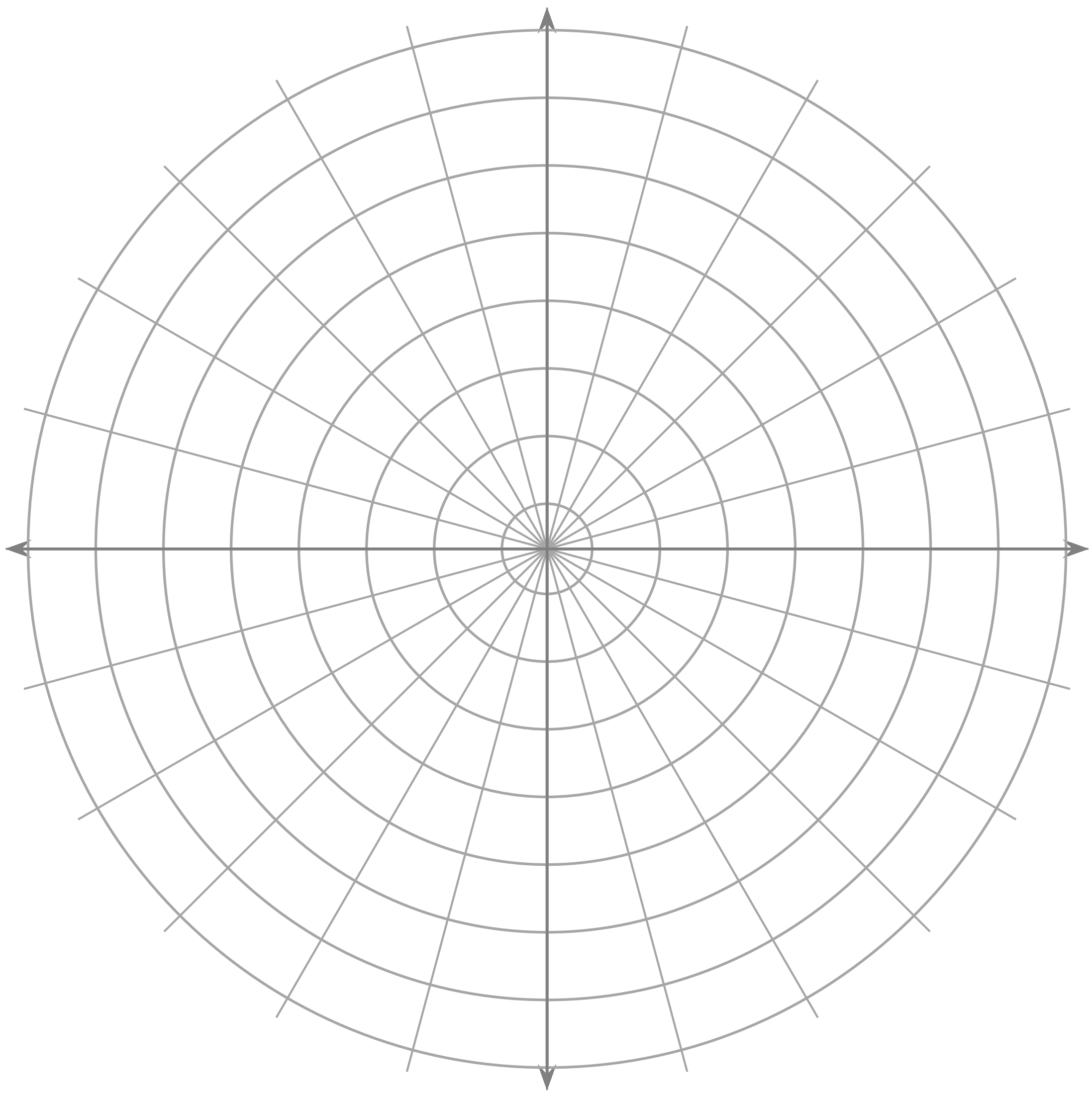
A quick review. We recall the following definition from Math 133.

Definition. Polar Coordinates

We fix a point called the **pole** (at the origin, O) and an **initial ray** from the pole (the positive x-axis). Then we locate each point P in the plane by the ordered pair (r, θ) where r is the directed distance from the origin and θ is the directed angle (or arc length, if we agree to use the unit circle) from the initial ray to the ray OP. It is customary to follow the usual conventions from trigonometry. In particular, counter-clockwise rotations define positive angles.



The following “polar-grid” should make it easier to plot points in polar coordinates.



Example 1. Plot each of the following polar coordinates in the xy-plane. Also, find all polar coordinate pairs for each of the given points.

a. $(3, 2\pi/3)$

For positive radii, we have

$$\left(3, \frac{2\pi}{3} + 2\pi \cdot k\right), \quad k \in \mathbb{Z}.$$

For negative radii,

$$\left(-3, \frac{-\pi}{3} + 2\pi \cdot k\right), \quad k \in \mathbb{Z}.$$

b. $(-2, \pi/4)$

We have

$$\left(-2, \frac{\pi}{4} + 2\pi \cdot k\right), \quad k \in \mathbb{Z}$$

and

$$\left(2, \frac{5\pi}{4} + 2\pi \cdot k\right), \quad k \in \mathbb{Z}.$$

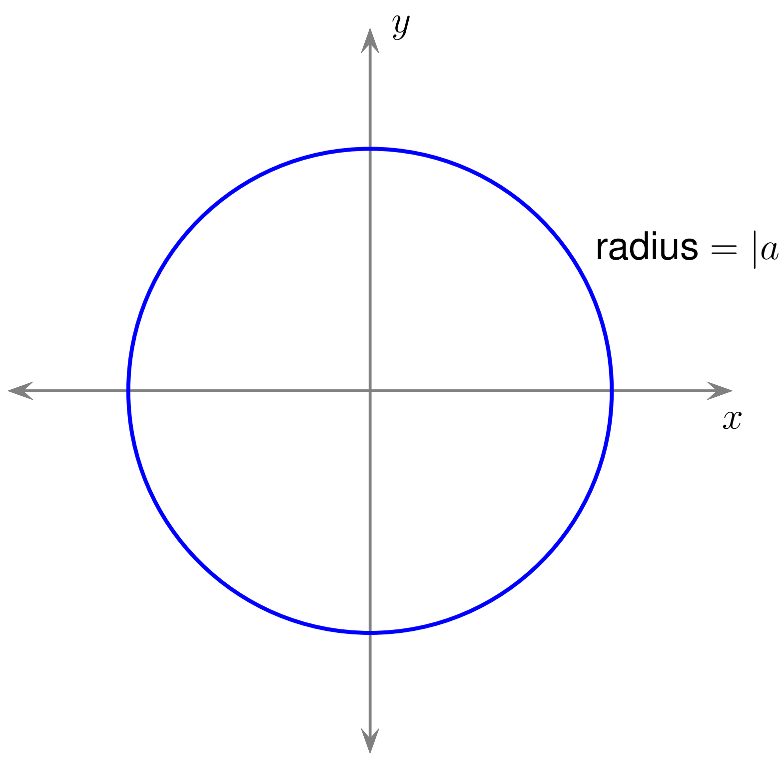
Polar Equations and Graphs

Let a and b be real constants.

In Cartesian coordinates, the basic equations $x = a$ and $y = b$ generate, respectively, vertical and horizontal lines in the xy-plane.

Example 2. Circles

The polar equation $r = \pm a$ generates a circle of radius $|a|$ in the xy-plane. (Why?)



Polar and Rectangular Coordinates

From elementary trigonometry we have the following equations that allow one to convert from polar to rectangular or from rectangular to polar coordinates (and hence, equations).

(1) $$x = r\cos\theta, \qquad y = r\sin\theta$$

(2) $$x^2 + y^2 = r^2, \qquad \tan\theta = \frac{y}{x}$$

Example 3. Rectangular to Polar

Find the equivalent equation in polar coordinates and sketch the graph.

a. $x^2 + y^2 = 9$.

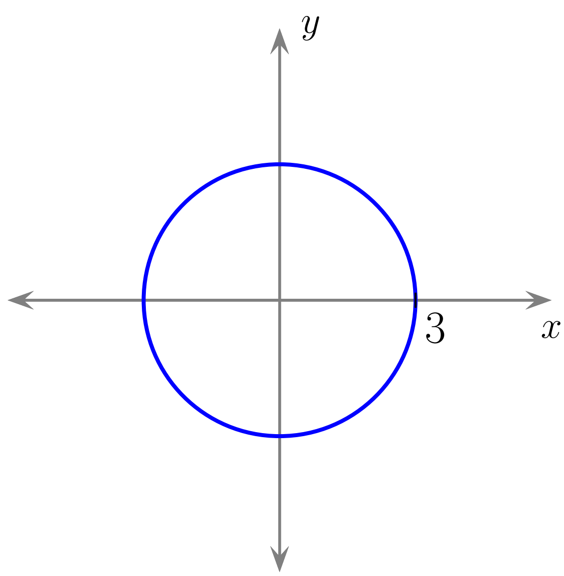
From (2) we have

$$x^2 + y^2 = r^2 = 9$$

or

$$r = 3$$

So we have a circle of radius 3 centered at the origin.



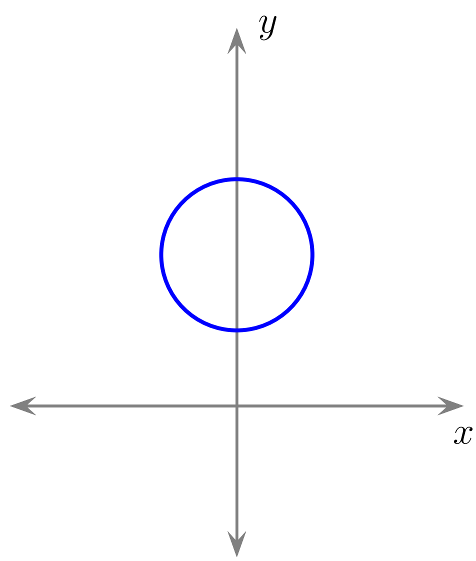
b. $x^2 + y^2 - 4y + 3 = 0$

First, we complete the square.

$$x^2 + y^2 - 4y + 4 = 1$$
$$x^2 + (y-2)^2 = 1^2$$

So, this is a circle of radius 1 centered at $(0, 2)$. Now, (2) implies

$$r^2 - 4r\sin\theta = -3$$



Example 4. Polar to Rectangular

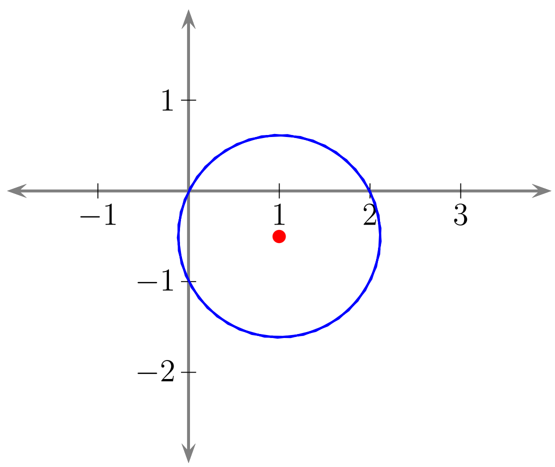
Find the equivalent equation in rectangular coordinates and sketch the graph.

$$r = 2\cos\theta - \sin\theta$$

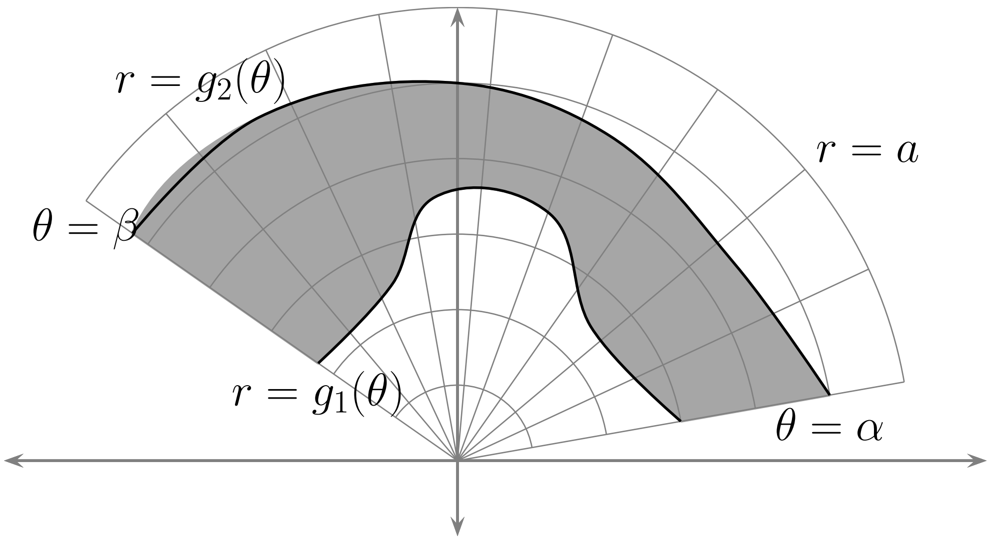
Multiplying both sides by r, we obtain

$$r^2 = 2r\cos\theta - r\sin\theta$$
$$x^2 + y^2 = 2x - y$$

It is now easy to see that this is the equation of a circle.



Double Integrals



Suppose that $f(r, \theta)$ is defined over the shaded region R shown in the sketch. Here the continuous curves g_1 and g_2 satisfy the inequality

$$0 \leq g_1(\theta) \leq g_2(\theta) \leq a$$

for $\alpha \leq \theta \leq \beta$.

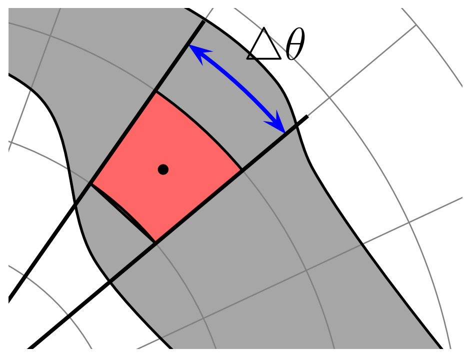
As before, we form the (Riemann) Sum

(3) $$S_n = \sum_{k=1}^{n} f\left(r_k, \theta_k\right) \, \Delta A_k$$

If f is continuous on R then the sum in (3) will approach a limit as ΔA_k goes to 0. Thus

(4) $$\lim_{\Delta A_k \to 0} S_n = \iint_R f(r, \theta)\, dA$$

As usual, we call this limit the double integral of f over R. However, we have another problem. What is dA?



Recall that the area of a circle is $\pi r^2 = 2\pi r^2/2$. It follows that the area of a sector with central angle θ is given by

$$A_{\text{S}} = \frac{\theta r^2}{2}$$

It follows that the area of the (differential) region shown above is

$$\begin{aligned}\triangle A &= \frac{\triangle\theta}{2}\left[r_2^2 - r_1^2\right]\\ &= \frac{\triangle\theta}{2}\left[\left(r_k + \frac{\triangle r}{2}\right)^2 - \left(r_k - \frac{\triangle r}{2}\right)^2\right]\\ &= \frac{\triangle\theta}{2}\left[\cancel{r_k^2} + r_k\triangle r + \cancel{\frac{\triangle r^2}{4}} - \cancel{r_k^2} + r_k\triangle r - \cancel{\frac{\triangle r^2}{4}}\right]\\ &= 2\,\left(\frac{r_k}{2}\right)\triangle r\,\triangle\theta\end{aligned}$$

so that

$$dA = r\,dr\,d\theta$$

Now (4) can also be written as

(5) $$\iint_R f(r,\theta)\,dA = \iint_R f(r,\theta)\,r\,dr\,d\theta$$

Theorem 1. Fubini's Theorem in Polar Coordinates

(6) $$\iint_R f(r,\theta)\,dA = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\,r\,dr\,d\theta$$

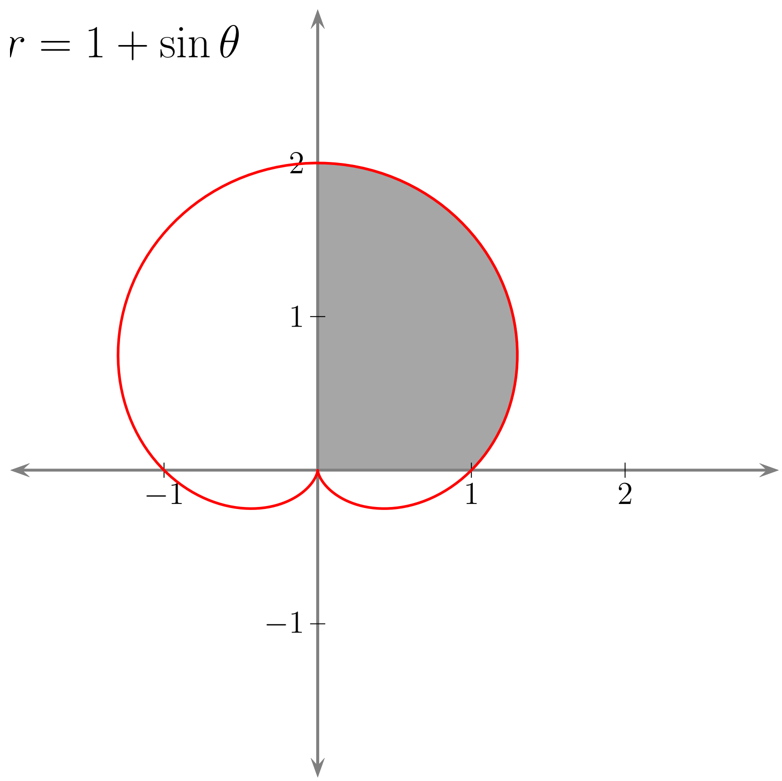
From this we immediately get

Area in Polar Coordinates

(7) $$A = \int\int_R r\,dr\,d\theta$$

Example 5. Computing Areas in Polar Coordinates

Let R be the shaded region in the sketch below.



a. Find the area of the shaded region.

By equation (7) and Fubini's Theorem

$$\begin{aligned} A &= \iint_R r\,dr\,d\theta \\ &= \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=1+\sin\theta} r\,dr\,d\theta \\ &= \dots \end{aligned}$$

b. Let $f(r,\theta) = r\cos\theta$ and compute the double integral of f over the given region.

$$\begin{aligned} \iint_R f(r,\theta)\,dA &= \iint_R f(r,\theta)\,r\,dr\,d\theta \\ &= \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=1+\sin\theta} r^2\cos\theta\,dr\,d\theta \\ &= \dots \end{aligned}$$

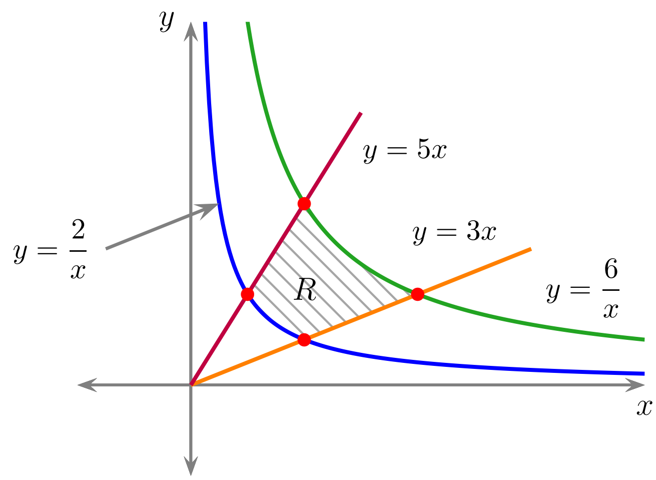


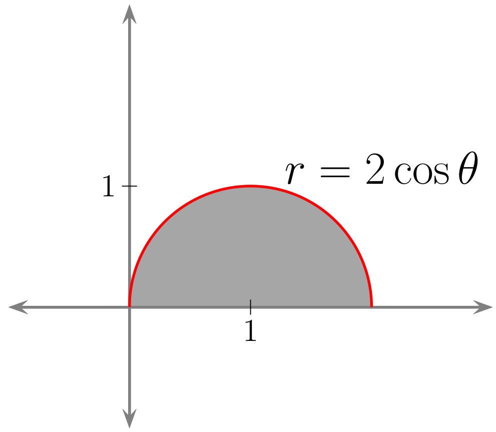
Figure 1: Region R from Example 6 (not to scale)

Example 6. In section 15.10, we found the area of the region R in Figure 1 by a change of variables. This time we will try switching to polar coordinates.

$$\begin{aligned}\text{Area of } R &= \iint_R dA = \iint_S r\,dr\,d\theta \\ &= \int_{\tan^{-1}3}^{\tan^{-1}5} \int_{\sqrt{\frac{4}{\sin 2\theta}}}^{\sqrt{\frac{12}{\sin 2\theta}}} r\,dr\,d\theta \\ &= \frac{1}{2}\int_{\tan^{-1}3}^{\tan^{-1}5} \frac{12}{\sin 2\theta} - \frac{4}{\sin 2\theta}\,d\theta \\ &= 4\int_{\tan^{-1}3}^{\tan^{-1}5} \csc 2\theta\,d\theta \\ &= -2\ln(\csc 2t + \cot 2t)\,\Big|_{\tan^{-1}3}^{\tan^{-1}5} \\ &\vdots \\ &= -2\ln 3/5\end{aligned}$$

as we saw before.

Example 7. More Polar Integrals



a. Find the area of the shaded region above.

$$\begin{aligned}
A &= \iint_R dA \\
&= \int_0^{\pi/2} \int_{r=0}^{r=2\cos\theta} r\,dr\,d\theta \\
&= \frac{1}{2}\int_0^{\pi/2} (2\cos\theta)^2 d\theta \\
&= \int_0^{\pi/2} (1+\cos 2\theta) d\theta \\
&= \left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/2} = \frac{\pi}{2}
\end{aligned}$$

as expected.

b. Let $f(r,\theta) = r\sin\theta$. Find $\iint_R f\,dA$.

Proceeding much as did in the previous example, we have

$$\begin{aligned}
\iint_R f\,dA &= \int_0^{\pi/2} \int_{r=0}^{r=2\cos\theta} r^2 \sin\theta\,dr\,d\theta \\
&= \frac{1}{3}\int_0^{\pi/2} \sin\theta\, r^3 \Big|_{r=0}^{r=2\cos\theta} d\theta \\
&= \frac{8}{3}\int_0^{\pi/2} \sin\theta\ \cos^3\theta\,d\theta \\
&= \frac{8}{3}\int_0^{1} u^3\,du \\
&= \frac{2}{3}u^4 \Big|_0^1 \\
&= \frac{2}{3}
\end{aligned}$$

Example 8. Converting from Rectangular to Polar Coordinates

Evaluate the integrals below.

a. $\displaystyle\iint_{\text{Unit Disk}} \sqrt{x^2+y^2}\,dA$

Since the integrand has no elementary antiderivative with respect to x or y, we try polar coordinates. Thus

$$\begin{aligned}\iint_{\text{Unit Disk}} \sqrt{x^2+y^2}\,dA &= \int_0^{2\pi}\int_0^1 \sqrt{r^2}\,r\,dr\,d\theta \\ &= 2\pi\int_0^1 r^2\,dr \\ &= 2\pi\,\frac{r^3}{3}\,\bigg|_0^1 = \frac{2\pi}{3}\end{aligned}$$

b. $\displaystyle\int_0^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\left(1+x^2+y^2\right)^2}\,dy\,dx$

It is possible to evaluate **improper** double integrals using many of the same ideas from Calc II. Consider the following example (cf. exercise 15.4.40).

Example 9. Improper Double Integrals

Evaluate $I = \displaystyle\int_0^\infty e^{-x^2}\,dx$.

Let $f(x) = e^{-x^2}$ and let $g(x) = 1/x^2$. Since

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$$

we can use the Limit Comparison Test to conclude that $I < \infty$. We now use the following (clever) technique to compute I.

$$
\begin{aligned}
I^2 &= \left(\int_0^\infty e^{-x^2}\,dx\right)^2 \\
&= \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right) \\
&= \int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)}\,dx\,dy \\
&= \int_0^{\pi/2} \int_0^\infty e^{-r^2} r\,dr\,d\theta \\
&= \frac{\pi}{2}\int_0^\infty e^{-r^2} r\,dr \\
&= \frac{-\pi}{4} e^{-r^2}\,\Big|_0^\infty = \frac{-\pi}{4}(0-1) \\
&= \frac{\pi}{4}
\end{aligned}
$$

It follows that $I = \dfrac{\sqrt{\pi}}{2}$ and, by symmetry,

$$
\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}
$$

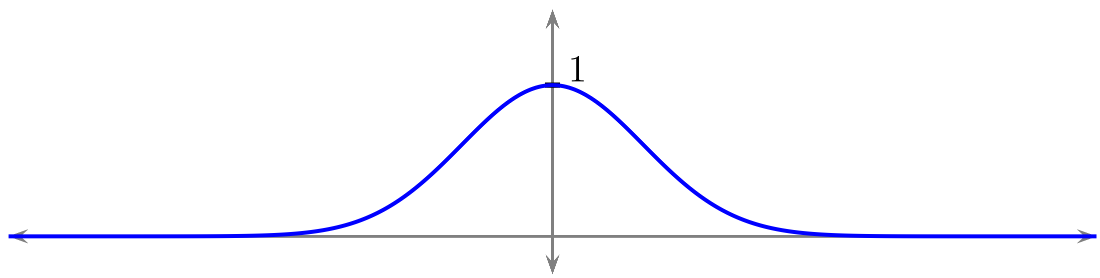
Remark. The integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

is related to the *normal* distribution function from probability and statistics. In fact, the function

$$f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

is the familiar "bell curve".

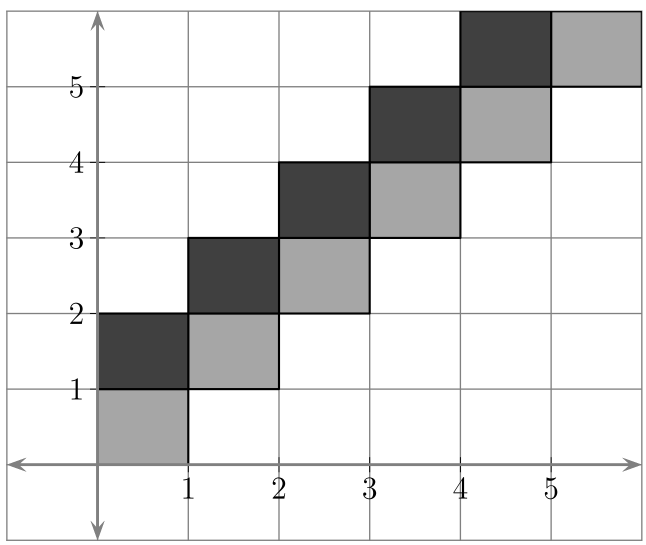


Can you evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx$?

Fubini's Theorem and Unbounded Regions

In section 15.1 Fubini's Theorem allowed us to switch the order of integration whenever it was convenient. The situation becomes more complex when the region R is unbounded. Consider the following example.

Example 10. Let R be the first quadrant and let $f(x, y)$ be defined as follows. If (x, y) is a point inside of a lightly shaded square (see the sketch below), then $f(x, y) = 1$. If (x, y) is a point inside of a darkly shaded square, then $f(x, y) = -1$. Otherwise, let $f(x, y) = 0$.



Now evaluate

$$\int_0^\infty \int_0^\infty f(x, y)\, dx\, dy$$

and

$$\int_0^\infty \int_0^\infty f(x, y)\, dy\, dx$$

Notice that

$$\int_0^\infty \int_0^\infty f(x,y)\,dx\,dy = 1 + (-1+1) + (-1+1) + \cdots$$

$$= 1 + 0 + 0 + \cdots = 1$$

and

$$\int_0^\infty \int_0^\infty f(x,y)\,dy\,dx = (-1+1) + (-1+1) + (-1+1) + \cdots$$

$$= 0 + 0 + 0 + \cdots = 0$$

We conclude that $\iint_R f(x,y)\,dA$ does not exist. Why?

Remark. The example above can be modified so that f is continuous with the same result. That is, the integral fails to exist.

It turns out that Fubini's Theorem needs to be used with care over *unbounded* regions. In fact, the pathologies in the previous example do not occur if $f(x,y) \geq 0$ over the given (unbounded) region.