### 15.1 Review of the Definite Integral

In a first semester calculus class, one eventually learns about something called a Riemann Sum

(1) $$S_P = \sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

where $f$ is a function defined over some finite interval $[a, b]$ with partition $P$.

It turned out to be important to investigate the following limit

$$\lim_{\|P\|\to 0} S_P = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

which is known to exist whenever $f$ satisfies certain properties (e.g., if it is continuous, etc.). In such cases we define the definite (or Riemann) integral of $f$ over the interval $[a, b]$ by

(2) $$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

It was later proven that the definite integral could be easily evaluated (in many cases) by using the **Fundamental Theorem of Calculus**. That is,

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F$ is any antiderivative of $f$.

The definite integral displayed several useful properties.

**Properties of the Definite Integrals**

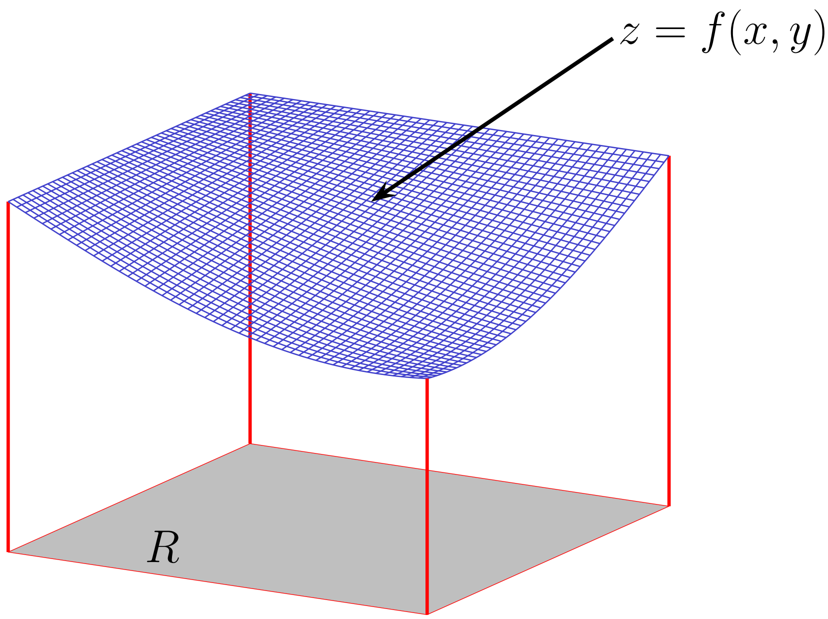
1. $\int_a^b k\, f(x)\, dx = k \int_a^b f(x)\, dx, \quad k \in \mathbb{R}$
2. $\int_a^b \left(f(x) \pm g(x)\right)\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$
3. $f(x) \geq 0 \quad \Longrightarrow \quad \int_a^b f(x)\, dx \geq 0$
4. $f(x) \geq g(x) \quad \Longrightarrow \quad \int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx$
5. $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^a f(x)\, dx$

The first two properties follow immediately from the definition (2). (In higher level mathematics, we say the integral operator is “linear”.) Property 4 follows immediately from properties 1 and 3. Property 3 is also a direct consequence of the definition. It allowed one to define the so-called “area under the curve”.

**Definition.** Let $f$ be an integrable function on $[a, b]$. If $f(x) \geq 0$ for all $x \in [a, b]$, then we define the **area under the curve** $y = f(x)$ by

$$\text{Area} = \int_a^b f(x)\, dx$$

**Double Integrals**




Suppose (temporarily) that $f(x, y) \geq 0$ is defined over a rectangular region $R$ in the plane given by

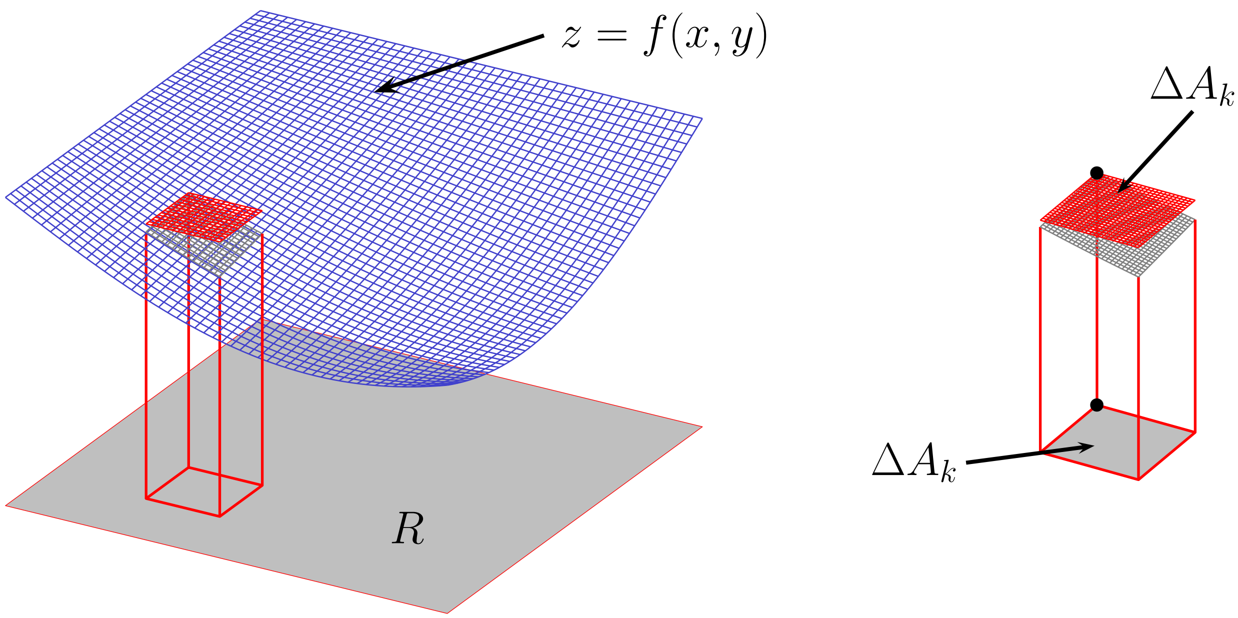
$$a \leq x \leq b$$
$$c \leq y \leq d$$

and suppose that wish to find the volume below the surface $z = f(x, y)$.

As we did in first semester calculus, we let $P_x$ and $P_y$ be partitions of $[a, b]$ and $[c, d]$ respectively. We may number these sub-rectangles, say $R_1,\ R_2,\ \ldots,\ R_n$. Now let $\Delta A_k$ be the area of rectangle $R_k$. Then

$$\Delta A_k = \Delta x\, \Delta y$$

Now we choose an arbitrary point $(x_k, y_k)$ in the rectangle $R_k$.




A typical rectangular solid is shown in the sketch above.

Now form the (Riemann) Sum

(3) $$S_n = \sum_{k=1}^{n} f\left(x_k, y_k\right)\, \Delta A_k$$

Now for any partition $P$ of rectangles, we define the **norm of** $P$, $\|P\|$, as the largest length or width of any of the rectangles. From our experience with area under a curve, we suspect that the quantity in (3) is a good approximation of the volume below the surface $z = f(x, y)$ whenever $\|P\|$ is sufficiently small.

We may then consider the following limit

(4) $$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f\left(x_k, y_k\right) \, \Delta A_k$$

Now if $f$ is continuous on $R$ (and in other cases) we know that the limit in (4) exists. The resulting limit is called the **double integral** and denoted by

$$\iint_R f(x,y)\,dA \quad \text{or} \quad \iint_R f(x,y)\,dx\,dy$$

It turns out that we can drop the requirement that $f(x,y) \geq 0$. We have

**Definition.** The double integral of $f$ over a rectangle $R$ is

(5) $$\iint_R f(x,y)\,dA \; = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f\left(x_k, y_k\right) \, \Delta A_k$$

provided the limit exists. Whenever the limit in (5) exists, we say that $f$ is (Riemann) integrable. So, for example, continuous functions are integrable.

**Theorem 1. Properties of Double Integrals**

1. $\iint_R k\, f(x,y)\, dA = k \iint_R f(x,y)\, dA, \qquad k \in \mathbb{R}$

2. $\iint_R (f \pm g)\ dA = \iint_R f\, dA \pm \iint_R g\, dA$

3. $f(x,y) \geq 0 \quad \Longrightarrow \quad \iint_R f(x,y)\, dA \geq 0$

4. $f(x,y) \geq g(x,y)$ on $R \quad \Longrightarrow \quad \iint_R f\, dA \geq \iint_R g\, dA$

5. If $R = R_1 \cup R_2$ where $R_1$ and $R_2$ are nonoverlapping rectangles then

$$\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA$$

As a consequence of item 3 above, we have

**Definition.** Let $f(x,y) \geq 0$ over a rectangle $R$. Then the volume below the surface $z = f(x,y)$ is

$$V = \iint_R f(x,y)\, dA \tag{6}$$

provided the integral exists.

**Example 1.** Let $R$ be the rectangle given by $0 \le x \le 1$ and $0 \le y \le 2$. Use Theorem 1 and (6) to evaluate the following double integrals.

a. $\iint_R 3\,dA$

b. $\iint_R x\,dA$

c. $\iint_R (3x + 2y)\,dA$

d. $\iint_R x^2\,dA$

e. $\iint_R xy\,dA$

A few sketches of $z = xy$.