16.3 Path Independence and Conservative Fields

Definition. Path Independence

Let $\mathbf{F}$ be a field defined on an open region D in space, and suppose that the (work) integral $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is the same for all paths from A to B (in D). Then the integral $\int \mathbf{F} \cdot d\mathbf{r}$ is **path independent** in D and the field $\mathbf{F}$ is **conservative** on D.

It turns out that a field $\mathbf{F}$ is conservative if and only if $\mathbf{F} = \nabla f$, that is, if and only if $\mathbf{F}$ is a gradient vector field for some differentiable function f.

Definition. Potential Functions

If $\mathbf{F}$ is a field defined on D and $\mathbf{F} = \nabla f$ for some scalar function f on D, then f is called a **potential function** for $\mathbf{F}$.

Some important assumptions:

1. All curves are piecewise smooth.
2. If $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ then $M,\ N,\ P$ have continuous first partials.
3. D is an open, connected region in space.

Theorem 1. The Fundamental Theorem of Line Integrals

1. Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ be a vector field with continuous components throughout an open connected region D in space. Then there exists a differentiable function f such that

$$\mathbf{F} = \nabla f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}$$

   if and only if for all points A and B in D the integral $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is path independent in D.

2. In this case

(1) $$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B \nabla f \cdot d\mathbf{r} = f(B) - f(A)$$

Example 1. Let $\mathbf{F}$ be the force field

$$\mathbf{F} = y \sin z \, \mathbf{i} + x \sin z \, \mathbf{j} + (xy \cos z + \sin z) \, \mathbf{k}$$

and let $A = (1, 1, \pi/6)$ and $B = (2, 3, \pi/2)$. Find the work done along the straight line connecting A to B.

Notice that $f(x, y, z) = xy \sin z - \cos z$ is a potential function for $\mathbf{F}$ since

$$\nabla f = y \sin z \, \mathbf{i} + x \sin z \, \mathbf{j} + (xy \cos z + \sin z) \, \mathbf{k}$$

In other words, the field $\mathbf{F}$ is conservative. So by equation (1)

$$\begin{aligned} \int_A^B \mathbf{F} \cdot d\mathbf{r} &= \int_A^B \nabla f \cdot d\mathbf{r} \\ &= f(B) - f(A) \\ &= f\,(2, 3, \pi/2) - f\,(1, 1, \pi/6) \\ &= 6 - \left(1/2 - \sqrt{3}/2\right) \\ &= 11/2 + \sqrt{3}/2 \end{aligned}$$

Remark. We will see how to find f below.

Notice that if $\mathbf{F}$ is conservative then the line integral around any closed curve is

$$\int_A^A \mathbf{F} \cdot d\mathbf{r} = \int_A^A \nabla f \cdot d\mathbf{r} = f(A) - f(A) = 0$$

Theorem 2.

The following statements are equivalent:

1. $\int \mathbf{F} \cdot d\mathbf{r} = 0$ around every closed curve in D.
2. The field $\mathbf{F}$ is conservative on D.

Definition. Del Notation and Curl

We define a new object...the “del” operator:

$$\nabla = \mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}$$

(This is just a convenient notation to help remember some formulas below.)

We also define the **curl** of the vector field $\mathbf{F}$ by

$$\begin{aligned}\text{curl}\,\mathbf{F} &= \nabla \times \mathbf{F} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} \\ &+ \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}\end{aligned}$$

Remark. We will discuss curl in more detail in the next section.

Example 2.

Find the curl of the vector field

$$\mathbf{F} = y \sin z \, \mathbf{i} + x \sin z \, \mathbf{j} + (xy \cos z + \sin z) \ \mathbf{k}$$

$$\text{curl}\, \mathbf{F} = \nabla \times \mathbf{F}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y \sin z & x \sin z & (xy \cos z + \sin z) \end{vmatrix}$$

$$= \left(\frac{\partial \left(xy \cos z + \sin z \right)}{\partial y} - \frac{\partial \left(x \sin z \right)}{\partial z} \right) \mathbf{i}$$

$$+ \left(\frac{\partial \left(y \sin z \right)}{\partial z} - \frac{\partial \left(xy \cos z + \sin z \right)}{\partial x} \right) \mathbf{j}$$

$$+ \left(\frac{\partial \left(x \sin z \right)}{\partial x} - \frac{\partial \left(y \sin z \right)}{\partial y} \right) \mathbf{k}$$

$$= (x \cos z - x \cos z) \, \mathbf{i} + (y \cos z - y \cos z) \, \mathbf{j}$$

$$+ (\sin z - \sin z) \, \mathbf{k}$$

$$= \mathbf{0}$$

The result from the last example was no coincidence. It turns out that a field is conservative if the curl is zero.

More precisely, suppose that

$$\mathbf{F} = M\left(x, y, z\right)\,\mathbf{i} + N\left(x, y, z\right)\,\mathbf{j} + P\left(x, y, z\right)\,\mathbf{k}$$

is a field whose component functions have continuous first partials and D is a simply connected region in space. Then $\mathbf{F}$ is conservative on D if and only if

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \mathbf{0}.$$

Example 3.

Show that $\mathbf{F} = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$ is conservative.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$= \mathbf{i}\left(\frac{\partial\left(3xy^2z^2\right)}{\partial y} - \frac{\partial\left(2xyz^3\right)}{\partial z}\right) - \mathbf{j}\left(\frac{\partial\left(3xy^2z^2\right)}{\partial x} - \frac{\partial\left(y^2z^3\right)}{\partial z}\right)$$

$$+ \mathbf{k}\left(\frac{\partial\left(2xyz^3\right)}{\partial x} - \frac{\partial\left(y^2z^3\right)}{\partial y}\right)$$

$$= \mathbf{i}\left(6xyz^2 - 6xyz^2\right) - \mathbf{j}\left(3y^2z^2 - 3y^2z^2\right) + \mathbf{k}\left(2yz^3 - 2yz^3\right)$$

$$= \mathbf{0}$$

Finding Potential Functions

Suppose that $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ is conservative. How do we find a function f such that $\nabla f = \mathbf{F}$? To find the function f, observe that it must satisfy the following partial differential equations (PDEs).

$$\frac{\partial f}{\partial x} = M, \quad \frac{\partial f}{\partial y} = N, \quad \frac{\partial f}{\partial z} = P \tag{2}$$

We illustrate below.

Example 4.

Find the potential function f from Example 1.

Recall that

$$\mathbf{F} = y\sin z\,\mathbf{i} + x\sin z\,\mathbf{j} + (xy\cos z + \sin z)\,\mathbf{k}.$$

So by (2) we must solve the following partial differential equations, simultaneously.

$$\frac{\partial f}{\partial x} = y\sin z \tag{3}$$

$$\frac{\partial f}{\partial y} = x\sin z \tag{4}$$

$$\frac{\partial f}{\partial z} = xy\cos z + \sin z \tag{5}$$

Antidifferentiating (3) with respect to x yields

$$f(x,y,z) = xy\sin z + C_1$$

where C_1 does not depend on x. Thus

$$f(x,y,z) = xy\sin z + g(y,z)$$

for some differentiable function g. We now have a “candidate” function to work. Specifically, f must satisfy the remaining partial differential equations, (4) and (5).

Now (4) implies

$$x\sin z = \frac{\partial(xy\sin z + g(y,z))}{\partial y} = x\sin z + \frac{\partial g}{\partial y}$$

It follows that g does not depend on y. In other words,

$$f(x,y,z) = xy\sin z + h(z)$$

for some differentiable function h. Finally, (5) implies

$$xy\cos z + \sin z = \frac{\partial(xy\sin z + h(z))}{\partial z} = xy\cos z + h'(z)$$

It follows that $h(z) = -\cos z + C$ and hence

$$f(x,y,z) = xy\sin z - \cos z + C,$$

Here C is an arbitrary constant. (*Note:* As usual, we chose to let $C = 0$ in Example 1).

Exact Differential Forms

Definition. The expression

$$M\,dx + N\,dy + P\,dz \tag{6}$$

is called a **differential form**. It is called **exact** on a region D if there is a real-valued function f defined on D such that

$$\underbrace{\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz}_{df} = M\,dx + N\,dy + P\,dz \tag{7}$$

Now if the differential form (6) is exact on a region D in space and f is a scalar function defined on D satisfying (7), and $A,\,B \in D$ then

$$\begin{aligned}\int_A^B M\,dx + N\,dy + P\,dz &= \int_A^B df \\ &= f(B) - f(A)\end{aligned}$$

as a direct consequence of the Fundamental Theorem of Line Integrals (Theorem 1).

Notice that equation (7) is equivalent to the statement that the field $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$ is conservative. In other words, the differential form (6) is exact if and only if there a real-valued function f defined on D such that

$$\underbrace{\frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}}_{\nabla f} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k} \tag{8}$$

Example 5.

Show that $y^2z^3\,dx + 2xyz^3\,dy + 3xy^2z^2\,dz$ is exact and compute the integral

$$\int_{(0,0,0)}^{(1,1/2,-3)} y^2z^3\,dx + 2xyz^3\,dy + 3xy^2z^2\,dz$$

Let $f(x,y,z) = xy^2z^3$. Then

$$df = y^2z^3\,dx + 2xyz^3\,dy + 3xy^2z^2\,dz$$

It follows that the given *form* is exact. Thus

$$\int_{(0,0,0)}^{(1,1/2,-3)} y^2z^3\,dx + 2xyz^3\,dy + 3xy^2z^2\,dz$$

$$= \int_{(0,0,0)}^{(1,1/2,-3)} df$$

$$= f(x,y,z)\ \Big|_{(0,0,0)}^{(1,1/2,-3)}$$

$$= f\,(1,1/2,-3) - f(0,0,0)$$

$$= (1)((1/2)^2)((-3)^3) = -27/4$$

What do conservative vector field look like?

We know that gradient vector fields are conservative. Consider the following example.

Example 6.

Let $f(x, y) = x^2 + xy$. The gradient field, $\nabla f = (2x + y)\,\mathbf{i} + x\,\mathbf{j}$, is shown in Figure 1. Now suppose that C is any smooth (simple, closed) curve in $\mathbb{R}^2$. Is it believable that the circulation integral $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$?

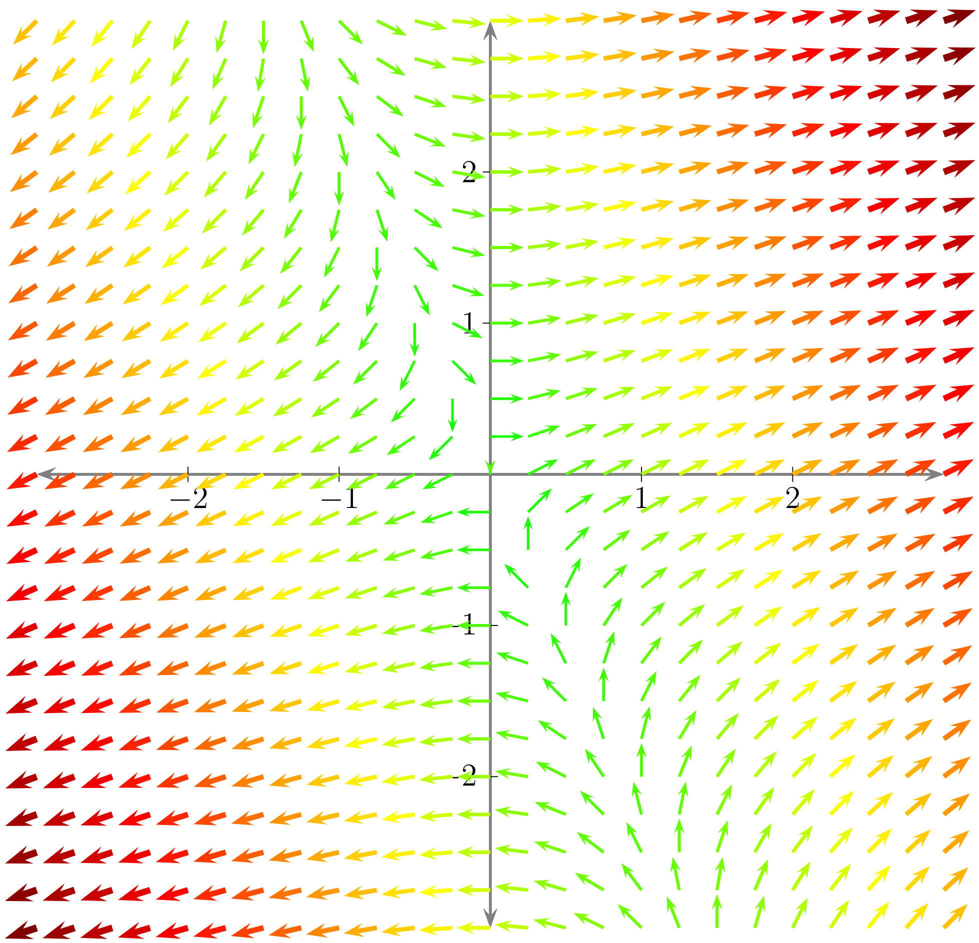


Figure 1: A Conservative Vector Field ∇f

We look at a few examples below.

(a) So let C be a circle of radius 2 centered at the origin. Verify that Theorem 2 holds.

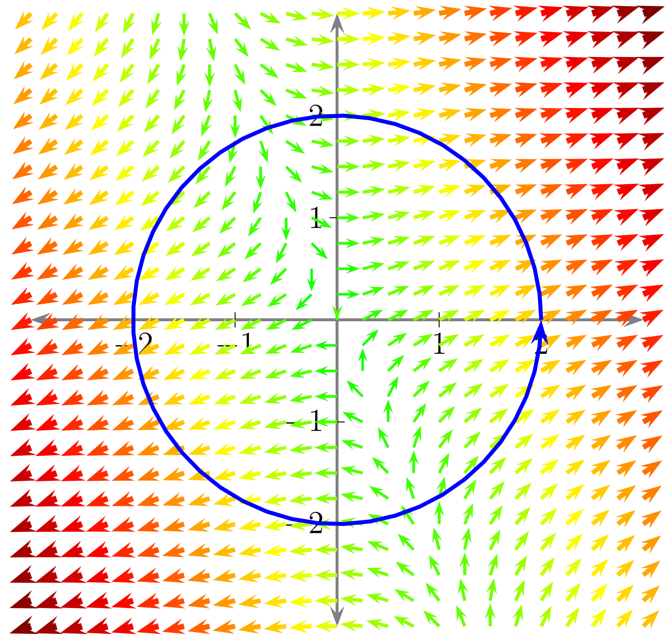
$$C\colon \mathbf{r}(t) = 2\,\cos t\,\mathbf{i} + 2\,\sin t\,\mathbf{j}, \quad 0 \le t \le 2\pi$$

$$M = 2x + y = 4\cos t + 2\sin t$$

$$N = x = 2\cos t$$

$$dx = -2\sin t\,dt$$

$$dy = 2\cos t\,dt$$



A direct calculation of the circulation integral yields

$$\oint_C M\,dx + N\,dy = \int_0^{2\pi} \left((4\cos t + 2\sin t)\,(-2\sin t) + (2\cos t)\,(2\cos t)\right)\,dt$$

$$= 4\int_0^{2\pi} \left(\cos^2 t - 2\sin t\,\cos t - \sin^2 t\right)\,dt$$

$$= 4\int_0^{2\pi} (\cos 2t - \sin 2t)\,dt$$

$$= 2\,(\sin 2t + \cos 2t)\,\Big|_0^{2\pi} = 0$$

as expected.

(b) Verify that Theorem 2 holds for the ellipse defined by

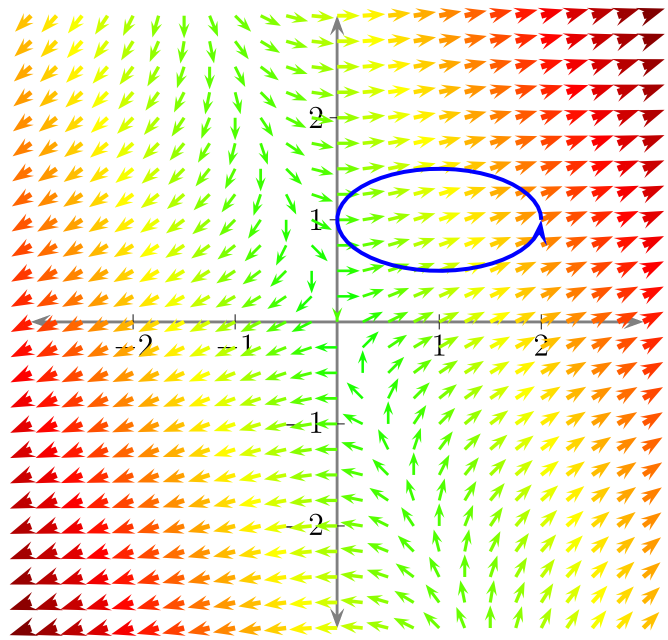
$$C\colon \mathbf{r}(t) = (1+\cos t)\,\mathbf{i} + \left(1 + \frac{\sin t}{2}\right)\mathbf{j}, \quad 0 \le t \le 2\pi$$

$$M = 2x + y = 3 + 2\cos t + \frac{\sin t}{2}$$

$$N = x = 1 + \cos t$$

$$dx = -\sin t\,dt$$

$$dy = \frac{\cos t}{2}\,dt$$



Once again a direct calculation of the circulation integral yields

$$\begin{aligned}\oint_C M\,dx + N\,dy &= \int_0^{2\pi} \left(\left(3 + 2\cos t + \frac{\sin t}{2}\right)(-\sin t) + (1+\cos t)\left(\frac{\cos t}{2}\right)\right) dt \\ &= \int_0^{2\pi} \left(-3\sin t - 2\sin t\cos t + \frac{\cos^2 t - \sin^2 t}{2} + \frac{\cos t}{2}\right) dt \\ &= \int_0^{2\pi} \left(-3\sin t - \sin 2t + \frac{\cos 2t}{2} + \frac{\cos t}{2}\right) dt \\ &= \vdots \\ &= 0\end{aligned}$$

as expected.

Remark. Of course, neither of the calculations above were necessary because, even if we hadn't noticed that the given field was conservative, we can simply check the $\mathbf{k}$-component of curl of $\mathbf{F}$. That is

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{\partial (x)}{\partial x} - \frac{\partial (2x+y)}{\partial y} = 1 - 1 = 0$$

Example 7. Evaluate the circulation integral $\oint_C (3y\,dx + 2x\,dy)$ where C is boundary of the region R defined by

$$0 \le x \le \pi,\ 0 \le y \le \sin x$$

Now let $\mathbf{F} = 3y\,\mathbf{i} + 2x\,\mathbf{j}$ (see Figure 2). We claim that $(3y)\,\mathbf{i} + (2x)\,\mathbf{j}$ is not exact.

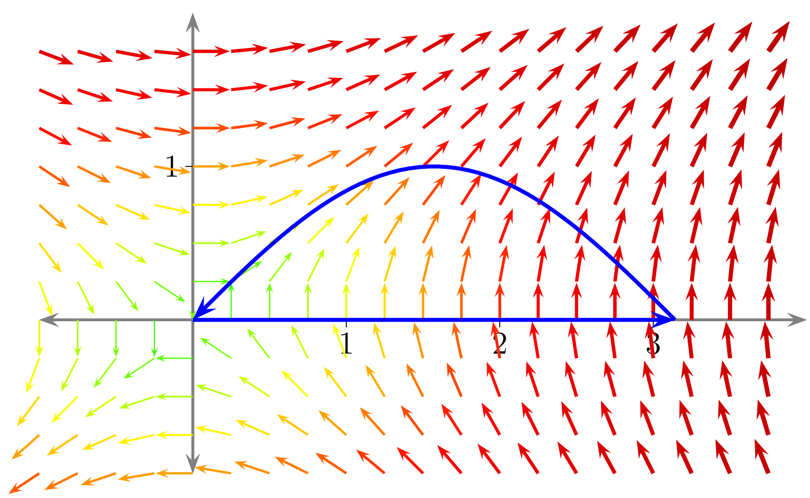


Figure 2: The field $\mathbf{F} = 3y\,\mathbf{i} + 2x\,\mathbf{j}$

Let's confirm by computing the circulation?

Let $C = C_1 \cup C_2$ where

$$C_1\colon \mathbf{r}_1(t) = (\pi - t)\,\mathbf{i} + \sin(\pi - t)\,\mathbf{j}, \quad 0 \le t \le \pi$$
$$C_2\colon \mathbf{r}_2(t) = t\,\mathbf{i}, \quad 0 \le t \le \pi$$

and

$$\mathbf{F}(\mathbf{r}_1(t)) = 3\sin(\pi - t)\,\mathbf{i} + 2(\pi - t)\,\mathbf{j}$$
$$\mathbf{F}(\mathbf{r}_2(t)) = 2t\,\mathbf{j}$$

Then

$$\oint_C (3y\,dx + 2x\,dy) = \int_{C_1} \mathbf{F}\cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}_2$$

Now

$$\begin{aligned}\int_{C_1} \mathbf{F}\cdot d\mathbf{r}_1 &= \int_0^\pi \langle 3\sin(\pi - t), 2(\pi - t)\rangle \cdot \langle -1, -\cos(\pi - t)\rangle\,dt \\ &= \int_0^\pi -3\sin(\pi - t) - 2(\pi - t)\cos(\pi - t)\,dt \\ &\vdots \\ &= -2\end{aligned}$$

and

$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r}_2 = 0$$

so that

$$\oint_C (3y\,dx + 2x\,dy) = -2 \ne 0$$

In other words, the $3y\,dx + 2x\,dy$ is not exact.