13.3 Arc Length

Last time we saw that the vector function $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ was differentiable at t if each of the components, $x(t)$, $y(t)$, and $z(t)$ were. Also, $\mathbf{r}$ was said to be **differentiable** if it was differentiable at each point in its domain. The curve traced by $\mathbf{r}$ is called **smooth** if $\dfrac{d\mathbf{r}}{dt}$ is continuous and never $\mathbf{0}$, i.e., if each of the component functions have continuous first derivatives that are not simultaneously 0.

Suppose that $C : \mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k},\ t \in [a,\, b]$ is **smooth** curve and that $P = P(t_0)$ is some base point on C. Then what is the distance, along the curve, $s(t)$ at any time $t > t_0$?

Whatever it is it must satisfy the differential equation

$$\frac{ds}{dt} = |\mathbf{v}(t)| \tag{1}$$

$$= |\mathbf{r}'(t)| \tag{2}$$

with the initial condition $s(t_0) = 0$.

That is, the **arc-length** parameter $s(t)$ is given by

(3) $$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)|\, d\tau$$

(4) $$= \int_{t_0}^{t} \sqrt{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2}\, d\tau$$

Notice that $s(t_0) = 0$ and $s(t) > 0$ whenever $t > t_0$.

This leads directly for the formula for the **length** of such a (smooth) curve.

Suppose that $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k},\ t \in [a,\, b]$ is a smooth curve traced exactly once as t increase from a to b. Then its **length** is given by

(5) $$L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Example 1. Arc Length

Consider the curve below.

$$\mathbf{r}(t) = \left(e^t\ \cos t\right)\ \mathbf{i} + \left(e^t\ \sin t\right)\ \mathbf{j} + e^t\,\mathbf{k}$$

a. Find the arc length parameter along the curve from the point where $t = 0$ by evaluating the integral

$$s = \int_0^t |\mathbf{v}(\tau)|\,d\tau$$

Now

$$\begin{aligned} \mathbf{v}(t) &= e^t(\cos t - \sin t)\ \mathbf{i} + e^t(\cos t + \sin t)\ \mathbf{j} + e^t\ \mathbf{k} \Longrightarrow \\ |\mathbf{v}(t)|^2 &= \left(e^t(\cos t - \sin t)\right)^2 + \left(e^t(\cos t + \sin t)\right)^2 + e^{2t} \\ &= 3e^{2t} \end{aligned}$$

Thus

$$\begin{aligned} s(t) &= \sqrt{3}\ \int_0^t e^\tau d\tau \\ &= \sqrt{3}\left(e^t - 1\right) \end{aligned}$$

b. Find the arc length of the curve over the interval $0 \le t \le \ln 4$.

From part (a) we see that the arc length is $s(\ln 4) = \sqrt{3}(4 - 1)$.

As we saw in (1),

$$\frac{ds}{dt} > 0$$

In other words, s is a one-to-one invertible function of t. It follows that t is also a differentiable function of s and

$$\frac{dt}{ds} = \frac{1}{ds/dt} = \frac{1}{|\mathbf{v}|}$$

Now by the chain rule we have

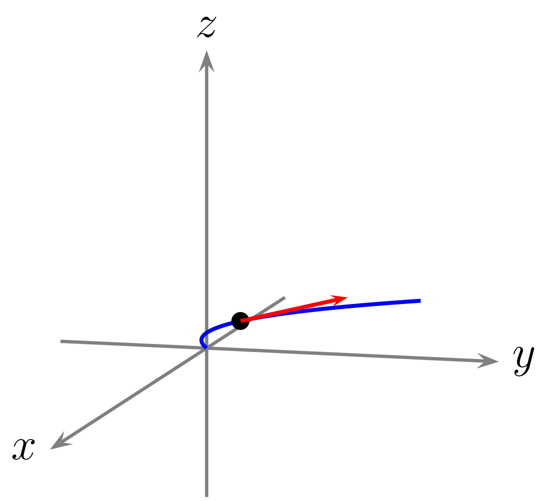
$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = \mathbf{v}\,\frac{1}{|\mathbf{v}|} = \frac{\mathbf{v}}{|\mathbf{v}|} \tag{6}$$

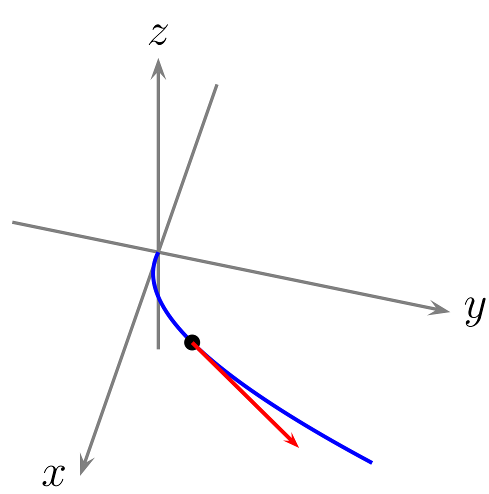
That is, $\mathbf{r}$ is a differentiable function of the arc-length parameter s. In fact, $\frac{d\mathbf{r}}{ds}$ is a unit vector in the direction of the velocity vector $\mathbf{v}$. We denote this vector by $\mathbf{T}$. It is called the unit tangent vector.

Definition

The **unit tangent vector** of a smooth curve $\mathbf{r}(t)$ is

(7) $$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|}$$





Example 2. The Unit Tangent Vector

The position function of a particle traveling in space is given below. Answer the questions that follow.

$$\mathbf{r}(t) = e^{-t}\,\mathbf{i} + 2\cos 3t\,\mathbf{j} + 3\sin 2t\,\mathbf{k}$$

a. Find $d\mathbf{r}/dt$.

$$\frac{d\mathbf{r}}{dt} = \left(-e^{-t}\right)\ \mathbf{i} + \left(-6\sin 3t\right)\ \mathbf{j} + \left(6\cos 2t\right)\ \mathbf{k}$$

b. Find the particle's speed and direction of motion at time $t = 0$.

$$\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = -\mathbf{i} + 0\,\mathbf{j} + 6\,\mathbf{k}$$

$$\Longrightarrow \mathbf{v}(0) = \underbrace{\sqrt{37}}_{\text{speed}} \times \underbrace{\left(\frac{-1}{\sqrt{37}}\,\mathbf{i} + \frac{6}{\sqrt{37}}\,\mathbf{k}\right)}_{\text{direction}}$$

c. Find the unit tangent vector of the curve at $t = 0$.

$$\begin{aligned}\mathbf{T} &= \frac{\mathbf{v}}{|\mathbf{v}|} \\ &= \frac{-1}{\sqrt{37}}\,\mathbf{i} + \frac{6}{\sqrt{37}}\,\mathbf{k}\end{aligned}$$

d. Find the unit tangent vector of the curve.

From part (a) we have,

$$|\mathbf{v}|^2 = \left(-e^{-t}\right)^2 + (-6\sin 3t)^2 + (6\cos 2t)^2$$

so that

$$\begin{aligned}\mathbf{T} &= \frac{\mathbf{v}}{|\mathbf{v}|} \\ &= \frac{\left(-e^{-t}\right)\ \mathbf{i} + (-6\sin 3t)\ \mathbf{j} + (6\cos 2t)\ \mathbf{k}}{\sqrt{\left(-e^{-t}\right)^2 + (-6\sin 3t)^2 + (6\cos 2t)^2}}\end{aligned}$$