1. (2 points) Suppose that f is continuous function of x and that $f(2) = 3$. Which of the statements below *must* be true about the function

$$g(x) = \int_0^x f(t)\,dt$$

(a) g is a continuous function of x.

Solution:

Since f is continuous, g is differentiable (by the FTC). But the differentiability of g implies the continuity of g. Hence the statement is true.

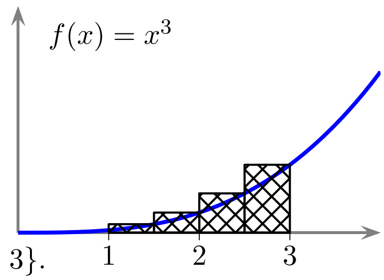
(b) $g'(2) = 3$.

Solution:

Since f is continuous the statement is true by the Fundamental Theorem of Calculus.

2. (4 points) *Estimate* the integral below by subdividing the appropriate interval into 4 equal subintervals and evaluating the corresponding Riemann sum using right end points. A SKETCH IS INCLUDED FOR YOUR CONVENIENCE.

$$\int_1^3 x^3\,dx$$



Solution:

(a) So $\Delta x = \dfrac{3-1}{4}$ and our partition is given by $P = \{1, 1.5, 2, 2.5, 3\}$.

(b) We compute the area of each rectangle by multiplying the base, $\Delta x = 0.5$ by the height, $f(x_j)$ (since we're using right end points). Thus

$$A_1 = f(x_1)\cdot\Delta x = (x_1)^3\cdot 0.5 = (1.5)^3\cdot 0.5$$
$$A_2 = f(x_2)\cdot\Delta x = \ldots$$

Notice that if we had been asked to compute the sums using **left** end points, we would have obtained the formula

$$A_j = f(x_{j-1})\cdot\Delta x$$

for the area of a typical rectangle.

(c) It follows that the Riemann sum is given by

$$\sum_{j=1}^{4} A_j = \sum_{j=1}^{4} f(x_j)\cdot\Delta x = \Delta x\cdot\sum_{j=1}^{4}(x_j)^3$$
$$= \frac{1}{2}\left(\left(\frac{3}{2}\right)^3 + (2)^3 + \left(\frac{5}{2}\right)^3 + (3)^3\right) = \frac{1}{2}\cdot 54$$

3. Use the (limit) definition of the integral to evaluate $\int_1^3 x^3\,dx$.

Solution:

(a) So $\Delta x = \dfrac{3-1}{n} = 2/n$ and our partition is now given by

$$P = \{1, 1+\frac{2}{n}, 1+\frac{2(2)}{n}, \ldots, 1+\frac{2(k)}{n}, \ldots, 1+\frac{2(n)}{n} = 3\}$$

(b) We compute the area of each rectangle by multiplying the base, $\Delta x = 2/n$ by the height, $f(x_j)$ (since we're using right end points). So the area of a typical rectangle is given by

$$\begin{aligned} A_k = f(x_k)\cdot\Delta x = (x_k)^3\cdot\frac{2}{n} &= \left(1+\frac{2(k)}{n}\right)^3\cdot\frac{2}{n} \\ &= \frac{2}{n}\left(1+\frac{6k}{n}+\frac{12k^2}{n^2}+\frac{8k^3}{n^3}\right) \end{aligned}$$

(c) It follows that the Riemann Sum is given by

$$\begin{aligned} S_n = \sum_{k=1}^{n} A_k &= \frac{2}{n}\sum_{k=1}^{n}\left(1+\frac{6k}{n}+\frac{12k^2}{n^2}+\frac{8k^3}{n^3}\right) \\ &= \frac{2}{n}\sum_{k=1}^{n}1 + \frac{2}{n}\sum_{k=1}^{n}\frac{6k}{n} + \frac{2}{n}\sum_{k=1}^{n}\frac{12k^2}{n^2} + \frac{2}{n}\sum_{k=1}^{n}\frac{8k^3}{n^3} \\ &= \frac{2}{n}\cdot n + \frac{12}{n^2}\frac{n(n+1)}{2} + \frac{24}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{16}{n^4}\left(\frac{n(n+1)}{2}\right)^2 \\ &= 2 + 6\left(1+\frac{1}{n}\right) + 4\left(\frac{2n^2+3n+1}{n^2}\right) + 4\left(1+\frac{1}{n}\right)^2 \end{aligned}$$

(d) Now to compute the definite integral, we evaluate the following limit

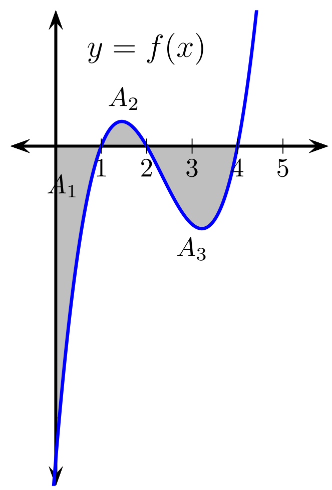
$$\begin{aligned} \lim_{n\to\infty} S_n = \lim_{n\to\infty} 2 + 6\left(1+\frac{1}{n}\right) + 4\left(\frac{2n^2+3n+1}{n^2}\right) + 4\left(1+\frac{1}{n}\right)^2 \\ = 2 + 6\,(1+0) + 4\,(2+0+0) + 4\,(1+0)^2 \\ = 2 + 6 + 4\cdot 2 + 4 = 20 \end{aligned}$$

In other words,

$$\int_1^3 x^3\,dx = 20$$

as we saw in class (using the Fundamental Theorem of Calculus).

4. (4 points) The graph of $y = f(x)$ is shown below. Answer the following questions. *Note:* The areas of the shaded regions are $A_1 = 3.25,\ A_2 = 0.5$ and $A_3 = 3$.



(a) Find the total area of the shaded regions.

Solution:

$A_1 + A_2 + A_3 = 6.75$

(b) Evaluate $\displaystyle\int_1^4 f(x)\,dx$

Solution:

$$\int_1^4 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^4 f(x)\,dx = 0.5 - 3$$

(c) Evaluate $\displaystyle\int_0^2 3f(x)\,dx - \int_2^4 f(x)\,dx$

Solution:

$$= 3\int_0^2 f(x)\,dx - (-3)$$
$$= 3(-3.25 + 0.5) + 3 = -5.25$$

(d) Evaluate $\displaystyle\int_0^4 |f(x)|\,dx$

Solution:

Should be the same as part (a).

$$= \int_0^1 (-f(x))\,dx + \int_1^2 f(x)\,dx + \int_2^4 (-f(x))\,dx$$
$$= -\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx - \int_2^4 f(x)\,dx = -(-3.25) + 0.5 - (-3) = 6.75$$