1. A rectangular metal box is to contain 36 cubic meters and will have a **square** base and top. The material for the base costs \$10 per square meter and the material for sides and top costs \$6 per square meter. Find the dimensions that minimize cost.

Solution:

(i) Minimize Cost

$$\begin{aligned} C &= C_{\text{base}} + C_{\text{Rest}} \\ &= 10x^2 + 6x^2 + 4 \times 6xy \end{aligned}$$

(ii) Constraints (volume)

Let the box dimensions be x by x by y. Then

$$\begin{aligned} x^2y = 36 &\implies \\ y = 36/x^2, &\quad x > 0 \end{aligned}$$

(iii) Combine steps (i) and (ii) to obtain a function of a single variable.

$$C(x) = 16x^2 + \frac{(24)(36)}{x}$$

(iv) Find the critical points. *Notice that it is much easier to take the derivative of this function as is (as opposed to combing the two rational functions).*

$$C'(x) = 32x - \frac{(24)(36)}{x^2}$$

and

$$C'(x) = 0 \quad \implies \quad x^3 = \frac{(24)(36)}{32} = 27$$

So the only critical point occurs at 3.

(v) Since our function is **not** defined on a closed interval we must find some other way to show that C attains a global minimum at 3 on the interval $(0, \infty)$. Notice that

$$C''(3) = 32 + \frac{2(24)(36)}{3^3} > 0$$

It follows that C has a local and hence global minimum at $x = 3$. See Figure 1. (*Note:* We could also construct a monotonicity chart.)

(vi) It follows from part (v) (and part (ii)) that the cost is minimized by the dimensions $3 \times 3 \times 4$.

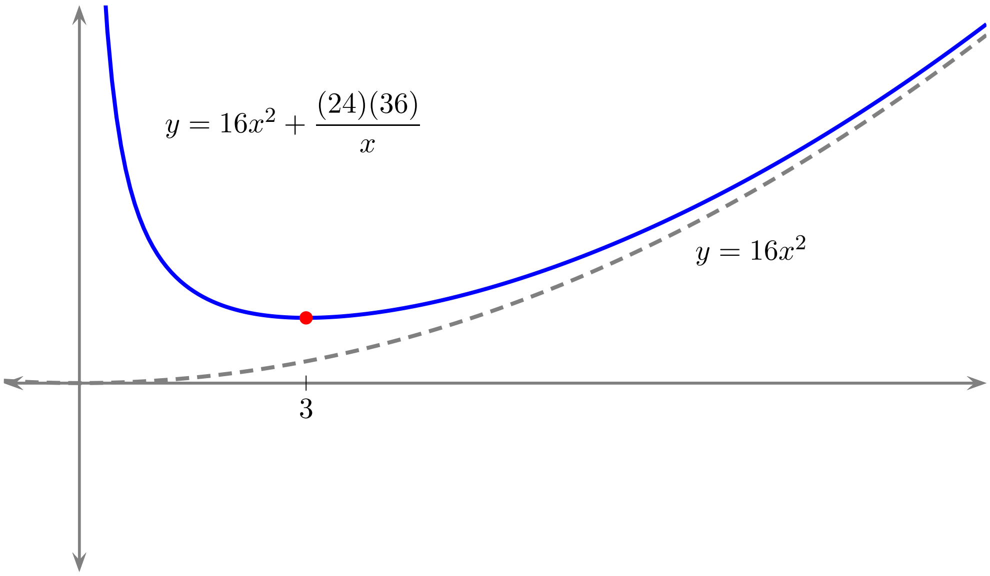


Figure 1: The Cost Function, $C(x)$.

It is worthwhile to note that $C(x)$ is asymptotic to $16x^2$ as $x \to \infty$. Using the language of limits, one would say that

$$\lim_{x\to\infty} \frac{C(x)}{16x^2} = 1$$

2. Suppose that a homeowner wishes to run power from the utility pole to the house as shown in the sketch below. A contractor charges \$8 per foot for overhead installation (along the property line) and \$10 per foot for underground installation (once inside the property). What is the minimum installation cost to supply power to the house? *Note: One possible path is shown in blue.*

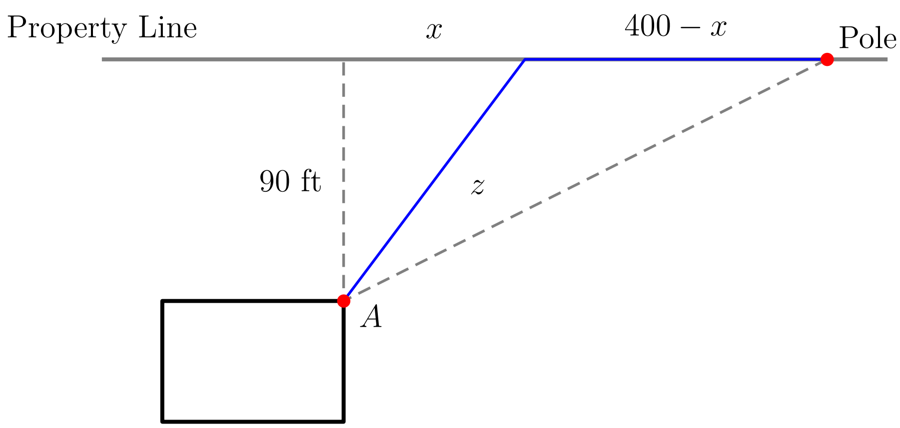


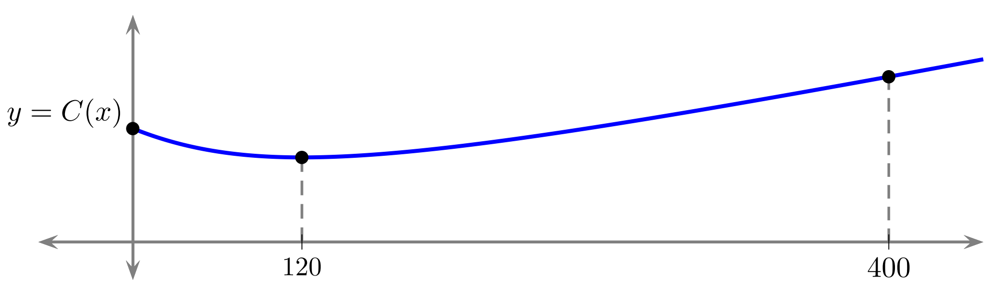
Figure 2: Supplying Residential Power

(a) **Identify Quantity to be Optimized**
We wish to minimize cost, $C = C_{\text{Underground}} + C_{\text{Overhead}}$. Notice that $C_{\text{U}} = 10z$ and $C_{\text{O}} = 8(400 - x)$

(b) **What are the restrictions?**
x and z are related by the equation $x^2 + 90^2 = z^2$ with $0 \leq x \leq 400$. Why?

(c) **Combining the items above we must minimize the function**

$$C(x) = 8(400 - x) + 10\sqrt{90^2 + x^2}, \quad x \in [0, 400]$$



(d) **Find the critical points of the function above.** Also, see the graph of the cost function above.

$$
\begin{aligned}
C'(x) &= \frac{10x}{\sqrt{x^2+90^2}} - 8 \\
C'(x) &= 0 \Longrightarrow \\
\frac{5x}{4} &= \sqrt{x^2+90^2} \\
\frac{25x^2}{16} &= x^2+90^2 \\
x &= 120
\end{aligned}
$$

(e) **Compare function values at the critical points and end points.**

x	$C(x)$
0	4100
120	3740
300	4100

(f) **Answer the question.**

By inspecting the table above we see that the minimum cost is $3740.