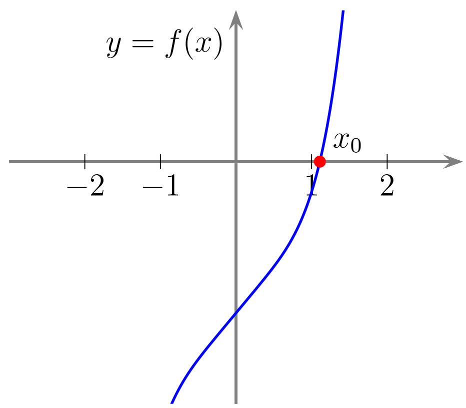
1.8 Continuity and the Intermediate Value Theorem

The general formula to solve a quadratic equation (i.e., a polynomial equation of degree 2) is well known. Some students are surprised to discover that there are also formulas to solve polynomial equations of degree 3 and 4. This section is motivated by the following example.

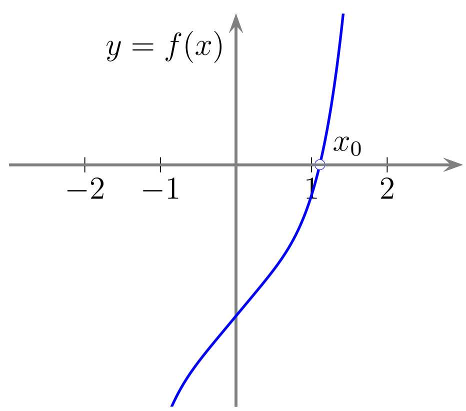
Example 1. In 1823 Niels Abel proved that there is no general formula to solve polynomial equations of degree 5 and higher. On the other hand, modern graphing calculators can be used to "solve" the following equation.

(1) $$x^5 + 3x - 5 = 0$$

If we use a utility to sketch the graph of $y = f(x) = x^5 + 3x - 5$, we see that the function appears to have a **zero**, say x_0, near 1.



That is, there appears to be a real number x_0 such that $f(x_0) = 0$. Is this true? How can we be sure that the function doesn't "jump" over the x-axis as shown in the sketch below?



Continuity

Definition. Continuity at a Point

Let $y = f(x)$ be a function defined on some set D and suppose that c is an interior point of D. That is, $c \in (a, b) \subset D$. We say that the f is **continuous at** c if

$$\lim_{x \to c} f(x) = f(c)$$

Note: See the text for the corresponding definitions if c is an *endpoint*.

Remark.

a. A function is simply called **continuous** if it is continuous at each point in its domain.

b. If f is not continuous at c, we say that f is **discontinuous** at c and that c is a **point of discontinuity** of f.

Example 2. Polynomials are continuous for all real numbers since $\lim_{x \to c} p(x) = p(c)$. On the other hand, let

$$g(x) = \frac{x^3 + 3x + 2}{x - 3}. \tag{2}$$

Then g is continuous on its domain. The real number 3, even though it is not in the domain of g, is a point of discontinuity of g.

Recall the (slightly modified form of the) limit laws from section 2.2.

Theorem 1. Suppose that $L, M, c,$ and k are real numbers and that

$$\lim_{x \to c} f(x) = f(c) \qquad \text{and} \qquad \lim_{x \to c} g(x) = g(c).$$

In other words, suppose that f and g are continuous at c.

a. *The Sum/Difference Rule*:

$$\lim_{x \to c} (f(x) \pm g(x)) = f(c) \pm g(c)$$

That is, the sum or difference of two continuous functions is a continuous function.

b. *The Product Rule*:

$$\lim_{x \to c} f(x) \cdot g(x) = f(c) \cdot g(c)$$

The product of two continuous functions is continuous.

c. *The Quotient Rule*:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}, \quad g(c) \neq 0$$

The quotient of two continuous functions is continuous.

d. *The Power Rule*: If r and s are integers with no common factor and $s \neq 0$, then

$$\lim_{x \to c} (f(x))^{r/s} = (f(c))^{r/s}$$

provided that the right-hand side is a real number. In other words, $f^{r/s}$ is also a continuous function.

The Power Rule suggests a more general rule.

Theorem 2. The composition of continuous functions is a continuous function.

Intermediate Value Property

Terminology: We say that f **attains** the value L if there is a c in the domain of f such that $f(c) = L$.

We return to Example 1. Pay close attention to the hypotheses in the following theorem.

Theorem 3. The Intermediate Value Theorem

Suppose f is a continuous function defined on the (closed) interval $[a, b]$. Then f attains every value between $f(a)$ and $f(b)$.

Example 3. Prove that the equation below has a *zero* between -3 and 2.

(3) $$x^5 + 3x - 5 = 0$$

Let $f(x) = x^5 + 3x - 5$. Then f is continuous on the interval $[0, 2]$ (Why?), $f(0) = -5$ and $f(2) = 33$. So by the Intermediate Value Theorem, there exists $c \in (0, 2) \subset (-3, 2)$ such that $f(c) = 0$.