1. Suppose that $0 < a < b$. Prove that there is a point $c \in (a, b)$ such that $c^2 = ab$. (*Hint:* There are at least two ways to proceed. **Method 1:** Let $f(x) = x^2 - ab$ show that $f(x)$ has a zero, etc. **Method 2:** Let $g(x) = 1/x$ and use the MVT.)

   Solution:

   Method 1: Let $f(x) = x^2 - ab$. Then f is continuous on the interval $[a, b]$ with $f(a) = a^2 - ab = a(a - b) < 0$ and $f(b) = b^2 - ab = b(b - a) > 0$. It follows by the Intermediate Value Theorem that there is a $c \in (a, b)$ such that $f(c) = c^2 - ab = 0$.

   Method 2: Let $g(x) = 1/x$. Then g is continuous on the interval $[a, b]$ and differentiable on (a, b). So by the MVT, there is a point $c \in (a, b)$ such that

$$\begin{aligned} \frac{-1}{c^2} &= g'(c) \\ &= \frac{g(b) - g(a)}{b - a} \\ &= \frac{1/b - 1/a}{b - a} \\ &= \frac{-1}{ab} \end{aligned}$$

   as desired.

   Method 3: Since $ab > 0$ we may set $c = \sqrt{ab}$. Hence $c^2 = ab$. Now

$$0 < a^2 < ab = c^2 < b^2 \quad \Longrightarrow \quad a < c < b$$

2. Let $f(x) = \dfrac{x^3}{3} - x^2 - 3x + 2$. Find the absolute maximum of $f(x)$ on the interval [1, 10] and say where it is attained. **Justify your answer.**

   Solution:

$$f'(x) = x^2 - 2x - 3 = (x - 3)(x + 1)$$

   It follows that the critical points are

$$x = -1, 3$$

   Now we just choose the largest from the following three function values.

$$\begin{aligned} f(1) &= -5/3 \\ f(3) &= -7 \\ f(10) &= 616/3 \end{aligned}$$

   So the absolute maximum is $f(10) = 616/3$.

3. (12 points) Solve the following initial value problem.

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \qquad y(9) = 0$$

Solution:

$$\begin{aligned} \int dy &= \int \frac{dx}{2\sqrt{x}} \\ y &= \sqrt{x} + C \end{aligned}$$

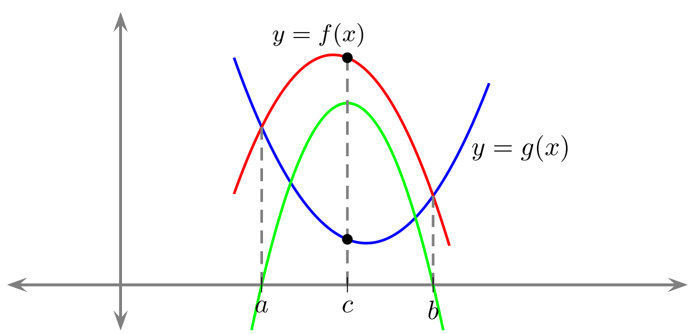
Now the initial conditions imply

$$0 = y(9) = \sqrt{9} + C \Longrightarrow C = -3$$

Thus

$$y(x) = \sqrt{x} - 3$$

4. Let $f(x)$ and $g(x)$ be differentiable functions that intersect at a and b (see the sketch). Suppose the vertical distance between the curves is **greatest** at $x = c$. Show that the tangent lines at $(c, f(c))$ and $(c, g(c))$ must be parallel.



Solution:

If the tangent lines are parallel, $f'(c) = g'(c)$. To see this, let

$$h(x) = f(x) - g(x)$$

The graph of $y = h(x)$ is shown in green. By assumption h has a local maximum at $x = c$. Since c is an interior point and h is differentiable at c, $h'(c) = 0$ by the First Derivative Theorem for Local Extreme Values. Thus

$$\begin{aligned} 0 &= h'(c) \\ &= f'(c) - g'(c) \end{aligned}$$

The result follows.

5. Evaluate the integrals.

   (a) $\displaystyle\int \left(5\sin 2x - 3\sec^2 x\right) dx$

   Solution:

$$= \frac{-5\cos 2x}{2} - 3\tan x + C$$

   (b) $\displaystyle\int \left(5x^3 - 2x^2 + \frac{4}{x^2}\right) dx$

   Solution:

$$= \frac{5x^4}{4} - \frac{2x^3}{3} - \frac{4}{x} + C$$

   (c) $\displaystyle\int \frac{x}{\sqrt{x+1}}\, dx$ (Hint: $x = x + 1 - 1$.)

   Solution:

   Observe that

$$\begin{aligned} \frac{x}{\sqrt{x+1}} &= \frac{x+1}{\sqrt{x+1}} - \frac{1}{\sqrt{x+1}} \\ &= \sqrt{x+1} - \frac{1}{\sqrt{x+1}} \end{aligned}$$

   so that

$$\begin{aligned} \int \frac{x}{\sqrt{x+1}}\, dx &= \int \left(\sqrt{x+1} - \frac{1}{\sqrt{x+1}}\right) dx \\ &= \frac{(x+1)^{3/2}}{3/2} - 2\sqrt{x+1} + C \end{aligned}$$

6. Let $g(x) = x(x-2)\sqrt{3-x}$. Answer the following questions.

   Note: $g'(x) = \dfrac{-5x^2+18x-12}{2\sqrt{3-x}}$ and $g''(x) = \dfrac{-3(x-2)(5x-16)}{4(3-x)^{3/2}}$.

   Note: The function is defined for $x \le 3$.

   (a) Identify the intervals on which g is increasing and decreasing. *You may use a monotonicity chart as we have done in class.*

   Solution:

   It is easy to see that the critical points are $a = \frac{9-\sqrt{21}}{5}$, $b = \frac{9+\sqrt{21}}{5}$. It follows that g is increasing on (a, b) and decreasing $(-\infty, a)$ and $(b, 3)$.

   (b) Identify the intervals on which g is concave up and concave down. *You may use a concavity chart as we have done in class.*

   Solution:

   The only possible inflection point occurs at 2. Notice that g is concave up on $(-\infty, 2)$ and concave down on $(2, 3)$.

   (c) Identify all local extrema. *Indicate whether the given point is a local maximum or minimum.*

   Solution:

   Since g is continuous for $x \le 3$, the monotonicity charts imply that g has local maximum at b and a local minimum at a.

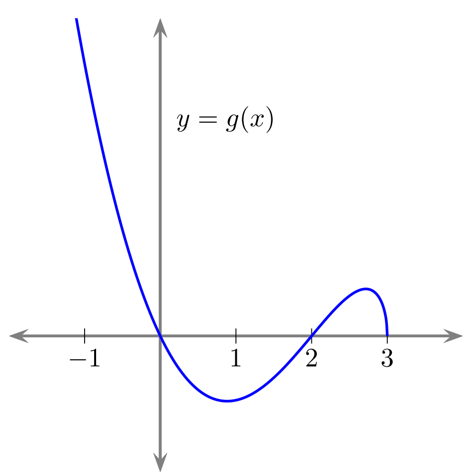
   (d) Identify all inflection points.

   Solution:

   There is a change in concavity at $x = 2$. Since g has a tangent line there, $x = 2$ is an inflection point.

   (e) Sketch the graph of $y = g(x)$. Give the coordinates of the local extremes.

   Notice that the function has zeros at $0, 2,$ and 3.



7. Let f be continuous on $[0, \infty)$ with $f(0) = 0$. Suppose that $f'(x) \geq 1$ for all $x \in (0, \infty)$. Prove that $f(x) \geq x$ for all $[0, \infty)$.

   Solution:

   One can prove this one using integration but the proof is awkward.

   Let $x > 0$. Then f satisfies the hypotheses of the MVT on the interval $[0, x]$. It follows that there is a $c \in (0, x)$ such that

$$\begin{aligned} \frac{f(x) - f(0)}{x - 0} &= f'(c) \geq 1 \\ \implies \frac{f(x)}{x} &\geq 1 \end{aligned}$$

   Combining this with the fact that $f(0) = 0$ yields the desired result.

8. Evaluate the following integrals.

   (a) $\displaystyle\int_3^5 (4x - 1)(x + 2)\, dx$

   Solution:

$$\begin{aligned} &= \int_3^5 (4x^2 + 7x - 2)\, dx \\ &= \left(\frac{4x^3}{3} + \frac{7x^2}{2} - 2x \right) \Bigg|_3^5 \\ &= \left(\frac{4(5)^3}{3} + \frac{7(5)^2}{2} - 2(5) \right) - \left(\frac{4(3)^3}{3} + \frac{7(3)^2}{2} - 2(3) \right) \\ &= \frac{548}{6} \end{aligned}$$

   (b) $\displaystyle\int_0^{\pi/3} \cos^2 x \, \sin x\, dx$

   Solution:

   We try u-substitution. Let $u = \cos x$. Then $du = -\sin x\, dx$ and $u(0) = 1,\ u(\pi/3) = 1/2$.

$$\begin{aligned} \int_0^{\pi/3} \cos^2 x \, \sin x\, dx &= -\int_1^{1/2} u^2\, du \\ &= \int_{1/2}^1 u^2\, du \\ &= \frac{u^3}{3} \Bigg|_{1/2}^1 = \frac{7}{24} \end{aligned}$$

(c) $\int 4t^2\sqrt{2+t^3}\,dt$

Solution:

$$\begin{aligned}\int 4t^2\sqrt{2+t^3}\,dt &= \frac{4}{3}\int \sqrt{u}\,du \\ &= \frac{8}{9}u^{\frac{3}{2}} + C \\ &= \frac{8}{9}(2+t^3)^{\frac{3}{2}} + C\end{aligned}$$

Once again you should check your work by taking the derivative of the above result.

9. Suppose that $f'(x) > 0$ for all x and that $f(1) = 0$. Which of the statements below *must* be true about the function

$$g(x) = \int_0^x f(t)\,dt$$

(a) g is a continuous function of x. **True**

(b) g is a differentiable function of x. **True**

(c) The graph of $y = g(x)$ has a horizontal tangent line at $x = 1$. **True**

(d) g has a local minimum at $x = 1$. **True**

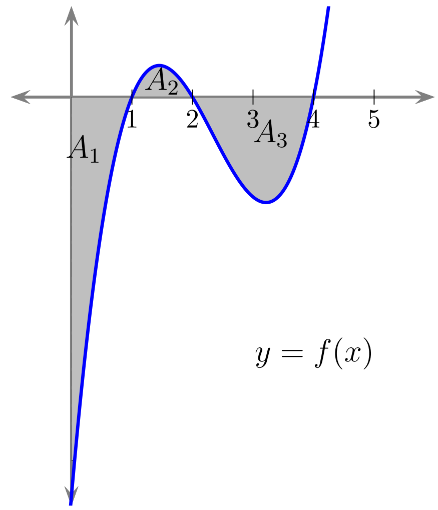
(e) g has a local maximum at $x = 1$. **False**

(f) The graph of $y = g(x)$ has an inflection point at $x = 1$. **False**

(g) The graph of $y = g'(x)$ crosses the x-axis at $x = 1$. **True**

(h) The graph of $y = g(x)$ is concave up (everywhere). **True**

10. The graph of $y = f(x)$ is shown below. Answer the following questions. *Note:* The areas of the shaded regions are $A_1 = 3.25,\ A_2 = 0.5$ and $A_3 = 3$.



(a) Find the total area of the shaded region.

$A_1 + A_2 + A_3 = 6.75$

(b) Evaluate $\int_1^4 f(x)\,dx$

$$\int_1^4 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^4 f(x)\,dx = 0.5 - 3$$

(c) Evaluate $\int_0^2 3f(x)\,dx - \int_2^4 f(x)\,dx = -5.25$

(d) What is the average value of f over the interval $[1, 4]$.

From part (b),

$$f_{\text{avg}} = \frac{1}{4-1}\int_1^4 f(x)\,dx = \frac{-5/2}{3}$$

11. Find dy/dx in each of the following.

(a) $y = \int_2^x t \sin t^2\,dt$

Solution:

By the FTC

$$\frac{dy}{dx} = x \sin x^2$$

(b) $y = \int_0^{\sqrt{x}} (1 + t^2)^6\,dt$

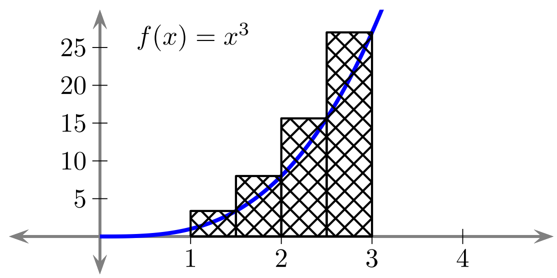
Solution:

By the FTC and the chain rule,

$$\frac{dy}{dx} = (1 + x)^6\ \frac{1}{2\sqrt{x}}$$

12. *Estimate* the integral below by subdividing the appropriate interval into 4 equal subintervals and evaluating the corresponding Riemann sum using right end points. A SKETCH IS INCLUDED FOR YOUR CONVENIENCE.

$$\int_1^3 x^3\,dx$$



Solution:

(a) So $\Delta x = \dfrac{3-1}{4}$ and our partition is given by $P = \{1,\ 1.5,\ 2,\ 2.5,\ 3\}$.

(b) We compute the area of each rectangle by multiplying the base, $\Delta x = 0.5$ by the height, $f(x_j)$ (since we're using right end points). Thus

$$A_1 = f(x_1)\cdot\Delta x = (x_1)^3\cdot 0.5 = (1.5)^3\cdot 0.5$$
$$A_2 = f(x_2)\cdot\Delta x = \ldots$$

Notice that if we had been asked to compute the sums using **left** end points, we would have obtained the formula

$$A_j = f(x_{j-1})\cdot\Delta x$$

for the area of a typical rectangle.

(c) It follows that the Riemann sum is given by

$$\begin{aligned}\sum_{j=1}^{4} A_j &= \sum_{j=1}^{4} f(x_j)\cdot\Delta x\\ &= \Delta x\cdot\sum_{j=1}^{4}(x_j)^3\\ &= \frac{1}{2}\left((x_1)^3+(x_2)^3+(x_3)^3+(x_4)^3\right)\\ &= \frac{1}{2}\left(\left(\frac{3}{2}\right)^3+(2)^3+\left(\frac{5}{2}\right)^3+(3)^3\right)\\ &= \frac{1}{2}\cdot 54\end{aligned}$$

13. In class we observed that $\sin x^2$ has no *elementary* antiderivative. Nevertheless, show that $\int_0^1 \sin x^2\, dx \le 1/3$. (*Hint:* You may freely use the fact that $x^2 - \sin x^2 \ge 0$ for all $x \in \mathbb{R}$.)

Solution:

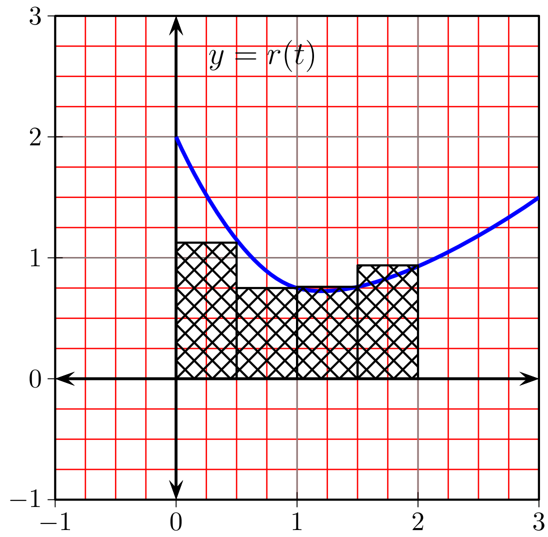
Recall that $f \ge 0 \implies \int f \ge 0$. Following the hint,

$$\int_0^1 \left(x^2 - \sin x^2\right)\, dx \ge 0$$

Rearranging the inequality above yields

$$\begin{aligned} \int_0^1 \sin x^2\, dx &\le \int_0^1 x^2\, dx \\ &= \frac{x^3}{3}\bigg|_0^1 = 1/3 \end{aligned}$$

14. The graph of a function $y = r(t)$ shows the rate of change (million bacteria/hour) in the population of a bacteria colony when the colony is treated by a certain drug. Answer the questions below.



(a) What are the units of $\int_0^2 r(t)\,dt$.

Number of bacteria (in millions).

(b) What is the practical meaning of $\int_0^2 r(t)\,dt$.

The change in the population of the colony after two hours.

(c) In the sketch, *carefully* shade in the precise region whose area is given by the **right-hand** sum with $n = 4$ (four subdivisions) for the definite integral $\int_0^2 r(t)\,dt$.

(d) Estimate $\int_0^2 r(t)\,dt$ by using the **right-hand** sum with $n = 4$.

From the sketch,

$$\int_0^2 r(t)\,dt \approx (1/2)\,(1.125 + 0.75 + 0.75 + 0.95)$$
$$= 1.7875$$

15. Use the following facts to answer the questions below. $\int_1^3 f(x)\,dx = 4,\ \int_1^3 g(x)\,dx = -6,$ and $\int_1^4 f(x)\,dx = 11$

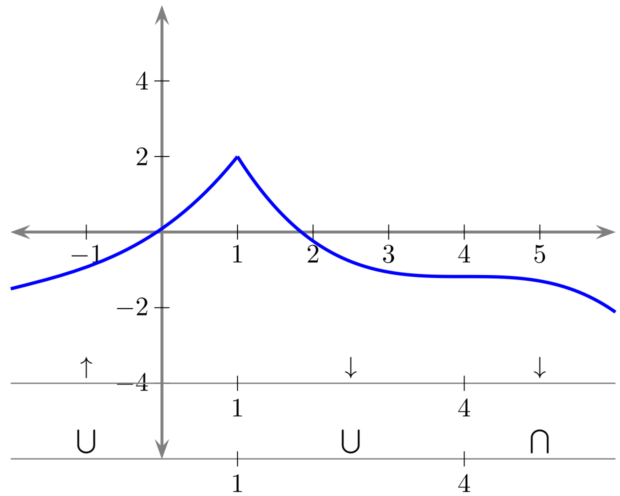
(a) $\int_1^3 (5f(x) - 2g(x))\,dx = 5(4) - 2(-6) = 32$

(b) $\int_3^4 f(x)\,dx = -4 + 11$

(c) If $H(x) = \int_1^x \frac{3^t}{\sqrt{t}}\,dt$, then $H'(x) = \frac{3^x}{\sqrt{x}}$

16. *Carefully* sketch the graph of a **continuous function** which has the following properties.

- $f(1) = 2$
- $f'(1)$ does not exist
- $f'(4) = f''(4) = 0$
- $f'(x) > 0$ for $x < 1$
- $f'(x) < 0$ for $1 < x < 4$ and $x > 4$
- $f''(x) > 0$ for $x < 1$ and $1 < x < 4$
- $f''(x) < 0$ for $x > 4$



17. Evaluate the integrals.

(a) $\displaystyle\int \left(\frac{\cos 3x}{4} + 5\sec 2x \tan 2x\right) dx = \frac{\sin 3x}{12} + \frac{5}{2}\sec 2x + C$

(b) $\displaystyle\int \left(5x^4 - 2x^3 - \frac{2}{\sqrt{3x}}\right) dx = x^5 - \frac{x^4}{2} - \frac{4\sqrt{3x}}{3} + C$

(c) $\displaystyle\int \sin x \cos x \, dx = \int \frac{\sin 2x}{2}\, dx = \frac{-\cos 2x}{4} + C \quad \text{or} \quad \frac{\sin^2 x}{2} + C \quad \text{or} \quad \frac{-\cos^2 x}{2} + C$

Can you explain why there are three antiderivatives? Does this violate Corollary 2 of the Mean Value Theorem (see section 4.2 of the text)?

18. Prove that $g(x) = 2\sqrt{x}$ is a *contraction mapping* on $[1, \infty)$. That is, show that if $a, b \in [1, \infty)$ then $|g(b) - g(a)| \leq |b - a|$. (*Hint:* $x \geq 1 \Longrightarrow \sqrt{x} \geq 1$.)

Solution:

Let $a, b \in [1, \infty]$ with $a < b$. Clearly g satisfies the hypotheses of the Mean Value Theorem on $[a, b]$. So there is a $c \in (a, b)$ such that

$$\begin{aligned}\frac{g(b) - g(a)}{b - a} &= g'(c) \\ &= \frac{1}{\sqrt{c}}\end{aligned}$$

Now taking absolute values of both sides yields

$$\begin{aligned}\left|\frac{g(b) - g(a)}{b - a}\right| &= \left|\frac{1}{\sqrt{c}}\right| \\ &= \frac{1}{\sqrt{c}} \\ &< 1\end{aligned}$$

where the last line is an immediate consequence of the the hint since $c > a \geq 1$. The result follows. (Notice that we have actually proved a stronger result...namely, the inequality is *strict.*)

19. Let $f(x) = \dfrac{x^3}{3} - x^2 - 2x$. Find the absolute **minimum** of $f(x)$ on the interval $[-2, 2]$ and say where it is attained. **Justify your answer.**

Solution:

$$f'(x) = x^2 - 2x - 2 = (x - 1)^2 - 3$$

It follows that the critical points are

$$x = 1 \pm \sqrt{3}$$

We choose the smallest from the following three function values...

$$\begin{aligned}f(-2) &= -8/3 \\ f\left(1 - \sqrt{3}\right) &= 2\sqrt{3} - \frac{8}{3} \\ f(2) &= -16/3\end{aligned}$$

So the absolute minimum is $f(2) = -16/3$.

20. Find at least one critical point of $p(x) = x\cos^2 x$.

Solution:

$$\begin{aligned}p'(x) &= \cos^2 x - 2x\cos x \sin x \\ &= \cos x(\cos x - 2\sin x)\end{aligned}$$

It follows that

$$p'(\pi/2) = 0$$

21. Let $g(x) = \dfrac{x^2+3}{x-1}$. Given that $g'(x) = \dfrac{(x+1)(x-3)}{(x-1)^2}$ and $g''(x) = \dfrac{8}{(x-1)^3}$, answer the questions below.

(a) Find the equation(s) of all asymptotes.

Solution:

Long division yields

$$g(x) = x + 1 + \frac{4}{x-1}$$

It follows that

V.A.: $x = 1$
I.A.: $y = x + 1$

(b) Identify the intervals on which g is increasing and decreasing. *You may use a monotonicity chart as we have done in class.*

Solution:

The critical points are $-1, 1,$ and 3. It is easy to show that g is increasing on $(-\infty, -1)$ and $(3, \infty)$ and decreasing $(-1, 1)$ and $(1, 3)$.

(c) Identify the intervals on which g is concave up and concave down. *You may use a concavity chart as we have done in class.*

Solution:

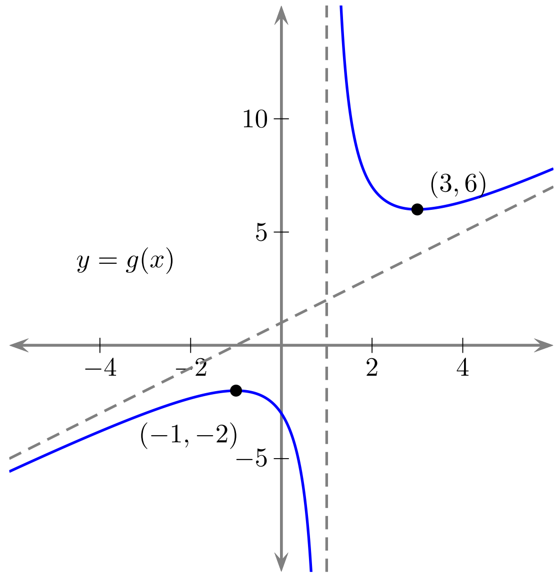
Notice that g is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$. Although g changes concavity at 1, there are no inflection points since $g(1)$ does not exist.

(d) Identify all local extrema. *Indicate whether the given point is a local maximum or minimum.*

Solution:

Since g is continuous at -1 and 3, the monotonicity chart imply that g has a local maximum at -1 and a local minimum at 3.

(e) Sketch the graph of $y = g(x)$. Give the coordinates of the local extremes. Use dashed lines to indicate the asymptotes.



22. Solve the following initial value problem.

$$\frac{dy}{dx} = \frac{2}{\sqrt{5x}}, \qquad y(5) = 0$$

Solution:

$$\int dy = \int \frac{2\,dx}{\sqrt{5x}}$$
$$y = \frac{4\sqrt{5x}}{5} + C$$

Now the initial conditions imply

$$0 = y(5) = \frac{4\sqrt{25}}{5} + C \Longrightarrow C = -4$$

Thus

$$y(x) = \frac{4\sqrt{5x}}{5} - 4$$

23. A rectangular metal box is to contain 36 cubic meters and will have a **square** base and top. The material for the base costs \$10 per square meter and the material for sides and top costs \$6 per square meter. Find the dimensions that minimize cost.

Solution:

(i) Minimize Cost

$$\begin{aligned} C &= C_{\text{base}} + C_{\text{Rest}} \\ &= 10x^2 + 6x^2 + 6(4)xy \end{aligned}$$

(ii) Constraints (volume)
Let the box dimensions be x by x by y. Then

$$\begin{aligned} x^2y &= 36 \implies \\ y &= 36/x^2, \quad x > 0 \end{aligned}$$

(iii) Combine steps (i) and (ii) to obtain a function of a single variable.

$$C(x) = 16x^2 + \frac{(24)(36)}{x}$$

(iv) Find the critical points.

$$C'(x) = 32x - \frac{(24)(36)}{x^2}$$

and

$$C'(x) = 0 \quad \implies \quad x^3 = \frac{(24)(36)}{32}$$

So the only critical point occurs at 3.

(v) Since our function is not defined on a closed interval we must find some other way to show that C attains a global minimum at 3 on the interval $(0, \infty)$. Notice that

$$C''(3) = 32 + \frac{2(24)(36)}{3^3} > 0$$

It follows that C has a local and hence global minimum at $x = 3$.

(vi) It follows from (v) that the cost is minimized by the dimensions $3 \times 3 \times 4$.

24. Suppose that g is continuous on $[a, b]$ and differentiable on (a, b) and that $g(a) > g(b)$. Show that $g'(x)$ is negative at some $x \in (a, b)$.

Solution:

By assumption g satisfies the hypotheses of the MVT. So there is a $c \in (a, b)$ such that

$$g'(c) = \frac{g(b) - g(a)}{b - a} = \frac{-}{+} < 0.$$