1. Show that a quadratic polynomial can have at most two real zeros.

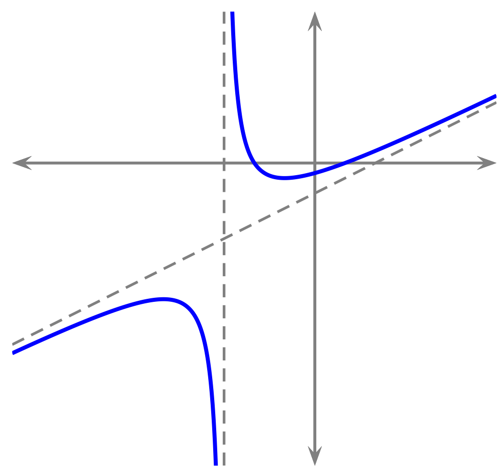
Solution:

Let p be be a quadratic polynomial. Then $p(x) = ax^2 + bx + c$ for real numbers $a, b,$ and c, $a \neq 0$. Observe that $p'(x) = 2ax + b$ has exactly one real zero, $-b/2a$. Now suppose that

$$p(x_1) = p(x_2) = p(x_3) = 0$$

for distinct real numbers, say $x_1 < x_2 < x_3$. Since p is differentiable everywhere, Rolle's Theorem implies that there are distinct real numbers s and t such that
$x_1 < s < x_2 < t < x_3$ with $p'(s) = p'(t) = 0$ contrary to the observation above. The result follows.

2. Let $f(x) = \dfrac{x^2 + x - 2}{x + 3}$ and answer the questions below. (*Note:* $f'(x) = \dfrac{x^2 + 6x + 5}{(x+3)^2}$ and $f''(x) = \dfrac{8}{(x+3)^3}$.)



(a) Find the zeros of $f(x)$.

$f(x) = \dfrac{(x+2)(x-1)}{x+3}$ implies that -2 and 1 are the zeros of f.

(b) Give the equation(s) of and label all asymptotes (vertical, horizontal, or inclined).

Write $f(x) = x - 2 + 4/(x+3)$. Then
V.A.: $x = -3$ I.A.: $y = x - 2$

(c) Identify the intervals on which f is increasing and decreasing. *You may use a monotonicity chart as we have done in class.*

The critical points of f are -5, -3, and -1. It is easy to see that, for example, $f' > 0$ on the interval $(-\infty, -5)$ hence f is increasing on $(-\infty, -5)$. It follows that f is increasing on $(-\infty, -5)$ and $(-1, \infty)$. f is decreasing on $(-5, -3)$ and $(-3, -1)$. Also, see sketch.

(d) Identify the intervals on which f is concave up and concave down. *You may use a concavity chart as we have done in class.*

The only possible inflection point is -3. Notice that $f'' < 0$ on $(-\infty, -3)$ hence f is concave down. Similarly, f is concave up on $(-3, \infty)$.

(e) Identify all local extremes. *Indicate whether the given point is a local maximum or minimum.*

f has a local maximum at $x = -5$ and a local minimum at $x = -1$.

(f) Identify all inflection points.

f has no inflection points since the only change in concavity occurs at $x = -3$ but f isn't even defined there.

(g) Sketch the graph of $y = f(x)$. Plot all the intercepts, critical points and possible inflection points and use *dashed* lines to indicate the asymptote(s).