WeBWork Homework Problem (Hw05 1.7 Limit Definition)

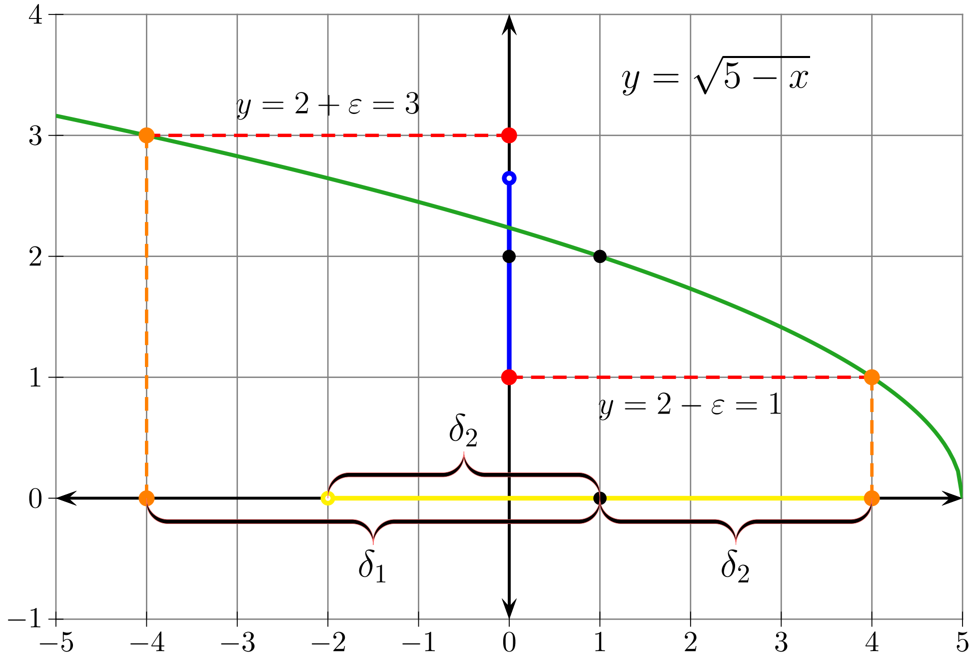

Figure 1: $\lim_{x\to 1} \sqrt{5-x} = 2$

It is a fact that $\lim_{x\to 1} \sqrt{5-x} = 2$. So according to the definition of the limit, given a (small) positive output tolerance $\varepsilon > 0$ we should be able to find an input tolerance $\delta > 0$ such that

$$0 < |x-1| < \delta \quad \text{implies} \quad |\sqrt{5-x} - 2| < \varepsilon \tag{1}$$

In Figure 1 we attempt to illustrate this by setting $\varepsilon = 1$. Notice that the input tolerance for $x < 1$ is more generous than the tolerance for $x > 1$. In particular, $\delta_1 > \delta_2$. This is not too surprising since the function is not linear so we should expect the deltas to be of different sizes.

It follows that if we choose $\delta = \min\{\delta_1, \delta_2\} = \delta_2 = 3$ then

$$0 < |x-1| < 3 \quad \text{implies} \quad |\sqrt{5-x} - 2| < 1$$

That is, if an input x falls within the yellow portion of the domain, then the output $y = \sqrt{5-x}$ will lie within the blue portion of the range!

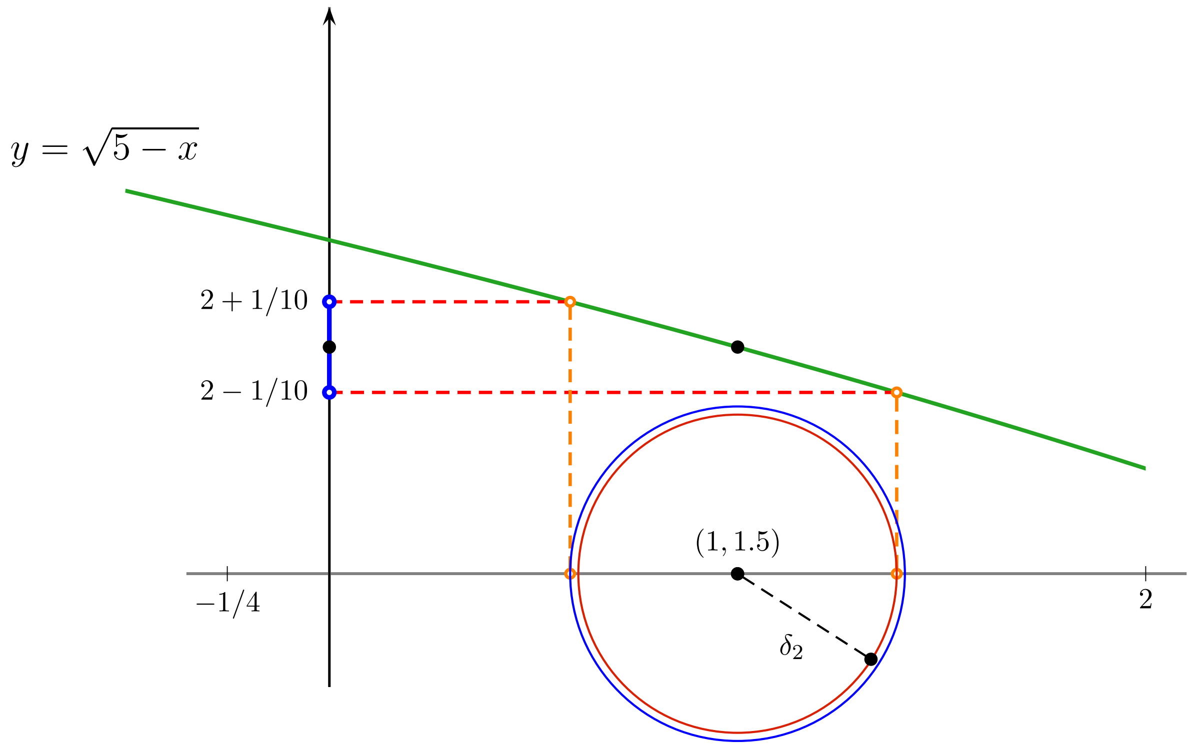

Figure 2: Zoomed in View of Figure 1 for $\varepsilon = 1/10$

Let's change the output tolerance from $\varepsilon = 1$ to $\varepsilon = 1/10$. In figure 2 we see a "close-up" shot of the relevant portion of the previous sketch.

Find $\delta = \delta(c, \varepsilon) > 0$ so that (1) holds. Recall that $\left|\sqrt{5-x} - 2\right| < 1/10$ is equivalent to

$$-1/10 < \sqrt{5-x} - 2 < 1/10$$

Now we solve for $x - 1$ (why?).

$$\begin{aligned}
19/10 &< \sqrt{5-x} < 21/10 \qquad \text{(See Figure 2)} \\
\implies (19/10)^2 &< 5 - x < (21/10)^2 \\
-(21/10)^2 &< x - 5 < -(19/10)^2 \\
\underbrace{4 - (21/10)^2}_{-\delta_1} &< x - 1 < \underbrace{4 - (19/10)^2}_{\delta_2}
\end{aligned}$$

It follows that

$$\delta_1 = \frac{41}{100} > \frac{39}{100} = \delta_2$$

A careful inspection of Figure 2 shows that $\delta_1 = 41/100$ is too big (see the blue circle).

So we let $\delta = \min\{\delta_1, \delta_2\} = \delta_2 = 39/100$, then

$$0 < |x - 1| < 39/100 \quad \text{implies} \quad |\sqrt{5-x} - 2| < 1/10$$