Example 3 The region \( R \) enclosed by curves \( y = \sqrt{x} \) and \( y = x^2 \) is rotated about the \( x \)-axis. Find the volume of the resulting solid.

A cylindrical cross-section is a circular washer.

\[
A(x) = \pi \left( \sqrt{x} \right)^2 - \pi (x^2)^2
= \pi (x - x^4)
\]

\[
V = \int_0^1 \pi (x-x^4) \, dx = \pi \left[ \frac{1}{2} x^2 - \frac{1}{5} x^5 \right]_0^1
= \pi \left[ \frac{1}{2} - \frac{1}{5} \right]
= \frac{3\pi}{10}
\]

So if cross section is a washer w/ some inner and outer radius,

\[
A = \pi \left[ \text{outer radius}^2 - \text{inner radius}^2 \right]
\]
Example 4. Find the volume obtained by rotating the region in
Example 3 about \( y = 2 \).

\[ A(x) = \pi \left[ (2-x)^2 - (2-\sqrt{x})^2 \right] \]
\[ = \pi \left[ 4 - 4x^2 + x^4 - (4 - 4\sqrt{x} + x) \right] \]
\[ = \pi \left[ x^4 - 4x^2 + x + 4\sqrt{x} \right] \]

\[ V = \int_0^1 A(x) \, dx \]
\[ = \int_0^1 \pi \left[ x^4 - 4x^2 + x + 4\sqrt{x} \right] \, dx \]
\[ = \pi \left[ \frac{1}{5}x^5 - \frac{4}{3}x^3 - \frac{1}{2}x^2 + \frac{8}{3}x^{3/2} \right]_0^1 \]
\[ = \pi \left[ \frac{1}{5} - \frac{4}{3} - \frac{1}{2} + \frac{8}{3} \right] \]
\[ = \frac{31}{30} \]
Example 5

Volume of solid obtained by revolving a region bounded by
\[
\begin{cases}
y = 1 + \sec x \\
y = 3
\end{cases}
\]

Points of intersection
\[
3 = 1 + \sec x \\
2 = \sec x \\
\frac{1}{2} = \cos x \\
-\frac{\pi}{3} \leq x \leq \frac{\pi}{3}
\]

Outer radius 2
Inner Radius \( \sec x \)

\[A(x) = \pi (2^2 - \sec^2 x)\]

\[
V = \int_{-\pi/3}^{\pi/3} \pi (4 - \sec^2 x) \, dx \\
= 2\pi \int_{0}^{\pi/3} (4 - \sec^2 x) \, dx \\
= 2\pi \left[ 4x - \tan x \right]_{0}^{\pi/3} \\
= 2\pi \left[ \frac{4\pi}{3} - \sqrt{3} \right]
\]
Quirky cross-sectional areas

Suppose we don't have a solid of revolution but we do have some data on what the cross-sections of a shape look like. Then we can still make some headway.

Example:

Find the volume of a solid with base a circle of radius 2 and cross-sections perpendicular to the base equilateral triangles.

So \( A(x) = \text{(area of triangle)} \)
\[
= \frac{1}{2} (2y) (\sqrt{3}y)
= \sqrt{3} y^2
= \sqrt{3} (4-x^2)
\]

\[
V = \int_{x=-2}^{x=2} \sqrt{3} (4-x^2) \, dx
= 2 \sqrt{3} \int_{0}^{2} (4-x^2) \, dx
= 2 \sqrt{3} \left( 4x - \frac{1}{3} x^3 \right)_{0}^{2}
= 2 \sqrt{3} \left( 8 - \frac{8}{3} \right)
= \frac{32}{3} \sqrt{3}
\]
Example 2

Find the volume of a pyramid whose base is a square w/ side L and whose height is h.

Doesn't come w/ coordinates, so we choose some.

Cross sections are squares of side lengths.

Looking at similar triangles in xy plane gives:

\[
\frac{\frac{L}{2}}{x} = \frac{\frac{L}{2}}{h}
\]

\[
S = \frac{Lx}{h}
\]

\[
A(x) = \frac{L^2}{h^2} x^2
\]

Volume = \[
\int_{x=0}^{x=h} \frac{L^2}{h^2} x^2 \, dx
\]

\[
= \frac{L^2}{h^2} \left[ \frac{1}{3} x^3 \right]_0^h
\]

\[
= \frac{L^2}{h^2} \left[ \frac{1}{3} h^3 \right]
\]

\[
= \frac{L^2 h}{3}
\]
Example 3 (More exciting)

A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder. Find the volume of the wedge.

\[
A(x) = \frac{1}{2} (y)(y \tan \frac{\pi}{3})
\]

\[
= \frac{1}{2} y^2 \left( \frac{1}{\sqrt{3}} \right)
\]

\[
= \frac{16 - x^2}{2 \sqrt{3}}
\]

\[
V = \int_{-4}^{4} \frac{16 - x^2}{2 \sqrt{3}} \, dx
= \frac{2}{2 \sqrt{3}} \int_{0}^{4} (16 - x^2) \, dx
\]

\[
= \frac{1}{\sqrt{3}} \left[ 16x - \frac{1}{3} x^3 \right]_{0}^{4}
\]

\[
= \frac{1}{\sqrt{3}} \left[ 64 - \frac{64}{3} \right]
\]

\[
= \frac{128}{3 \sqrt{3}}
\]
Problem: Find the volume obtained by rotating the region bounded by $y = 3x^2 - x^3$ and $y = 0$ about the $y$-axis.

Washer method is not great since we have to work out two strange functions of $y$, and also find maximum.

Since we can't approximate by cross-sections, we instead approximate by cylindrical shells.

Take a rectangle parallel to the axis and rotate it around the axis.

Volume cylinder = Volume outer cylinder - Volume inner cylinder

$$= \pi r_1^2 h - \pi r_2^2 h$$

$$= \pi h [r_1^2 - r_2^2]$$

$$= \pi h \left[ r_1 + r_2 \right] \left[ r_1 - r_2 \right]$$

$$= 2\pi h \left[ \frac{r_1 + r_2}{2} \right] \Delta r$$

Average radius of shell.
e.g. Volume = "circumference" * "thickness"

In our example: \( r = x_i \star \)
\( h = f(x_i \star) \)
\( \Delta r = Ax \)

\[
\text{Volume}_{\text{cylindrical shell}} = 2\pi \left( f(x_i \star) \right) x_i \star \Delta x
\]

\[
\text{Volume}_{\text{solid}} = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi \cdot f(x_i \star) x_i \star \Delta x
\]

\[
= \int_{0}^{2} 2\pi f(x) x \, dx
\]

\[
= 2\pi \int_{0}^{2} (2x^2 - x^3) \, dx
\]

\[
= 2\pi \left[ \frac{3}{4} x^4 - \frac{1}{5} x^5 \right]_0
\]

\[
= 2\pi \left[ \frac{243}{4} - \frac{243}{5} \right]
\]

\[
= 2\pi \frac{243}{20}
\]

\[
= \frac{243\pi}{10}
\]

So we can approximate a volume by taking a rectangle parallel to the axis of rotation, using it to generate a cylindrical shell whose volume is \( 2\pi r h \Delta x \), where \( r \) is the distance from the axis of rotation to the rectangle, \( h \) is the height of the rectangle (usually \( dx \) or \( dy \)).
2) Find the volume of the solid obtained by rotating the region bounded by \( xy = 1 \), \( x = 0 \), \( y = 1 \), and \( y = 3 \) about the \( x \)-axis.

\[
V_{\text{cylinder}} = 2\pi rh \tag{3}
\]

\[
= 2\pi \left[ \frac{y_i^*}{y_i} \right] \frac{1}{y_i} dy
\]

Volume \( = \lim_{n \to \infty} \sum_{i=0}^{n} 2\pi y_i^* \frac{1}{y_i} \Delta y \)

\[
= \int_{y=1}^{y=3} 2\pi y \, dy
\]

\[
= 2\pi \left[ \frac{y^2}{2} \right]_{1}^{3}
\]

\[
= 4\pi
\]
3. Find the volume of the solid generated by rotating the region bounded by $x = y^2 + 1$ and $x = 2$ about $y = -2$.

$$V_{cylinder} = 2\pi h r t = \frac{2\pi}{3} \int_{-1}^{1} (y^2 + 2 - y^3) dy$$

$$= 2\pi \int_{-1}^{1} \left[y + 2 - \frac{y^3}{3} - 2y^2\right] dy$$

$$= 2\pi \left[2 - \frac{2}{3} \right]$$

$$= \frac{16\pi}{3}$$
Volumes of solids

General

Place on axis, find cross-sectional area

Solids of Revolution

<table>
<thead>
<tr>
<th>Disk/Washer</th>
<th>$V = \int x^2 - r^2 , dx$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Approximate by spinning a rectangle perpendicular to axis of rotation</td>
<td></td>
</tr>
</tbody>
</table>

Cylindrical shells

Approximate by spinning a rectangle parallel to axis of rotation

$V = \int 2\pi rh \, dx$

$V = \int 2\pi rh \, dy$

Example 4

Set up integrals to find the volume of the solids given by rotating the region bounded by $y = -x^2 + 6x - 8$ and $y = 0$ about the $x$-axis and $y$-axis.

$0 = -x^2 + 6x - 8$
$0 = x^2 - 6x + 8$
$(x-4)(x-2)$

Vertex at 3

$-9 + 18 - 8 = 1$

$x$-axis: Disk method $r = y = -x^2 + 6x - 8$

$V = \int_2^4 \left(-x^2 + 6x - 8\right)^2 \, dx$
\( V = \int_2^4 2\pi x \left[ -x^2 + 6x - 8 \right] dx \)

y-axis: cylindrical shells \( r = x \) \( h = y = -x^2 + 6x - 8 \)