What is a sequence? A list of numbers.

\[ a_1, a_2, a_3, a_4, a_5, a_6, \ldots \]

Also written

\[ \sum_{n=1}^{\infty} a_n \text{ or } \sum_{n=1}^{\infty} a_n \]

- First term
- Second term
- Third term

Examples

(a) \[ \left\{ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots \right\} \quad a_n = \frac{1}{n} \]

(b) \[ \left\{ \frac{-2}{2}, \frac{4}{8}, \frac{-5}{16}, \ldots \right\} \quad a_n = \frac{(-1)^n (n+2)}{2^n} \]

(c) \[ \{2, \sqrt{5}, \sqrt{6}, \sqrt{7}, 3, \ldots \} \quad a_n = \sqrt{n-1} \]

(d) \[ \{0, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \ldots \} \quad a_n = \sin \left( \frac{n \pi}{6} \right) \]

Note that \( n \) doesn't have to start at 1.
Example

What sequence is

\[ a_1 = \frac{2}{3}, \quad a_2 = -\frac{3}{9}, \quad a_3 = \frac{y}{27}, \quad a_4 = \frac{-5}{81}, \quad a_5 = \frac{6}{243}, \ldots \]

\[ a_n = \frac{(-1)^n (n+1)}{3^n} \left\{ \frac{(-1)^{n+1} (n+1)}{3^n} \right\}_{n=1}^{\infty} \]

Not every sequence has a defining equation; some involve nonalgebraic data

Example

1. \( \{1, 2, 3, 5, 8, 13, 21, \ldots \} \quad \text{Fibonacci sequence} \)

Define recursively

\[ f_1 = 1 \]
\[ f_2 = 1 \]
\[ f_n = f_{n-1} + f_{n-2}, \quad n \geq 3 \]

2. \( \{3, 1, 4, 1, 5, 9, 2, 6, 5, \ldots \} \quad \text{Digits of \( \pi \)}\)

\[ a_n = \text{number in the \( n \)-th place} \]

3. \( \{59, 59, 42, 34, 18, 23, 18, 14, \ldots \} \)

What comes next? Christopher St. These are the stops on the \( \text{D} \) train below Columbus Circle.
What can we say about the behavior of a sequence $\{a_n\}$ as $n \to \infty$?

Example: $\{a_n = \frac{1}{n}\}$

Plot points $(n, a_n)$ on a coordinate axis.

As $n$ becomes large, $\frac{1}{n} \to 0$.

So it's reasonable to talk about a sequence having a limit.

**Defn** A sequence $\{a_n\}$ has the limit $L$ and we write

$$\lim_{n \to \infty} a_n = L \quad \text{or} \quad a_n \to L \quad \text{as} \quad n \to \infty$$

if we can make the terms $a_n$ as close to $L$ as we like by taking $n$ sufficiently large. If the limit exists, we say the series converges; if not we say it diverges.

Two sequences with limit $L$.

**Defn** A sequence $a_n$ has limit $L$ if $\forall \varepsilon > 0$ there is an $N$ such that if $n > N$ then $|a_n - L| < \varepsilon$. 
This looks almost identical to our previous definition of the limit of a function. We should exploit this!

\[ \lim_{n \to \infty} a_n = L \quad \text{and} \quad \lim_{x \to \infty} f(x) = L \]

The only difference is that \( x \) can be any number, but \( n \) has to be an integer.

"\( x \) is continuous, \( n \) is discrete."

Suppose \( \exists f \) st \( f(n) = a_n \).

Thm: If \( \lim_{x \to \infty} f(x) = L \) and \( f(n) = a_n \),

when \( n \) is an integer,

then \( \lim_{n \to \infty} a_n = L. \)

Examples

1. \( a_n = \frac{n}{n+1} \quad \lim_{n \to \infty} \frac{n}{n+1} = \lim_{x \to \infty} \frac{x}{x+1} = 1 \)

2. \( a_n = \frac{1}{n^2} \quad \lim_{n \to \infty} \frac{1}{n^2} = \lim_{x \to \infty} \frac{1}{x^2} = 0 \)

We also have the notion of an infinite limit.

\( \lim_{n \to \infty} a_n = \infty \) means that for every positive number \( M \) \( \exists \) an integer \( N \) such that if \( n > N \), \( a_n > M. \)
This is a special case of divergent sequences, in which the limit gets arbitrarily large.

Example

\[ s_1, 2, 3, 4, 5, 6, \ldots \]  
\[ a_n = n \]  
As \( n \to \infty \), \( a_n \to \infty \)

All the limit laws still hold by the same proofs as for functions.

If \( a_n \to L \) and \( b_n \to M \), are convergent, and \( c \in \mathbb{R} \),

1. \( \lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n \)
2. \( \lim_{n \to \infty} (a_n - b_n) = \lim_{n \to \infty} a_n - \lim_{n \to \infty} b_n \)
3. \( \lim_{n \to \infty} c a_n = c \lim_{n \to \infty} a_n \)
4. \( \lim_{n \to \infty} c = c \)
5. \( \lim_{n \to \infty} a_n b_n = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n \)
6. \( \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n} \) if \( \lim_{n \to \infty} b_n \neq 0 \)
7. \( \lim_{n \to \infty} a_n^p = \left[ \lim_{n \to \infty} a_n \right]^p \) if \( p > 0 \) and \( a_n > 0 \)
We also still have the Squeeze Theorem

If \( a_n \leq b_n \leq c_n \) for \( n \geq n_0 \) and \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L \), then \( \lim_{n \to \infty} b_n = L \).

One more useful fact.

**Theorem** If \( \lim_{n \to \infty} |a_n| = 0 \), then \( \lim_{n \to \infty} a_n = 0 \).

**Examples** Convergent or divergent?

1. \( a_n = \frac{n}{\sqrt{5+n}} \)

\[
\lim_{n \to \infty} \frac{n}{\sqrt{5+n}} = \lim_{n \to \infty} \frac{n}{\sqrt{\frac{5}{n^2} + \frac{1}{n}}} = \infty \quad \text{diverges to } \infty
\]
\[
\begin{align*}
2. \quad a_n &= \frac{\ln(n)}{n} \\
&= \lim_{n \to \infty} \frac{\ln(n)}{n} \\
&= \lim_{x \to \infty} \frac{\ln(x)}{x} \\
&= \lim_{x \to \infty} \frac{1}{x} \\
&= 0
\end{align*}
\]

Can't apply L'Hospital here because it pertains to functions.

3. \[a_n = (-1)^n, \quad n \geq 0\]

\[\{1, -1, 1, -1, 1, -1, \ldots\}\] diverges.

4. \[a_n = \frac{(-1)^n}{n}, \quad n \geq 1\]

Take the limit of the absolute value.

\[
\lim_{n \to \infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n \to \infty} \frac{1}{n} = 0
\]

Therefore \(\lim_{n \to \infty} \frac{(-1)^n}{n} = 0\) as well.

Note this only works if the limit is actually 0!
Notice: If the sequence is partly constructed from $c$-fts, we can use what we know about limits of $c$-ts functions to evaluate limits.

**Theorem:**

If $\lim_{n \to \infty} a_n = L$ and $f$ is $c$-ts at $L$, then $\lim_{n \to \infty} f(a_n) = f(L)$.

**Example:**

$$\lim_{n \to \infty} \cos \left( \frac{\pi}{n} \right) = \cos \left( \lim_{n \to \infty} \frac{\pi}{n} \right)$$

$$= \cos (0)$$

$$= 1$$

**Example:**

What about $a_n = \frac{n!}{n^n}$?

$$a_1 = 1$$

$$a_2 = \frac{1 \cdot 2}{2}$$

$$a_3 = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3}$$

$$a_n = \frac{1 \cdot 2 \cdot 3 \cdots \cdot n}{n \cdot n \cdot n \cdots n}$$

$$= \frac{1}{n} \left( \frac{2}{n} \cdots \frac{n}{n} \right)$$

$$\leq 1$$

We see:

$$0 \leq a_n \leq \frac{1}{n}$$

$$\Rightarrow \lim_{n \to \infty} a_n = 0 \text{ by the squeeze theorem.}$$
Example 3

\[ a_n = r^n \]

For what values of \( r \) does the series diverge?

If \(-1 < r < 1\)
\[
\lim_{n \to \infty} |r^n| = \lim_{n \to \infty} |r|^n = 0
\]

If \( r > 1 \)
\[
\lim_{n \to \infty} r^n = \infty
\]

If \( r < -1 \)
\[
|1| \to \infty
\]
Sign of \( r \) varies divergent

If \( r = 1 \)
\[
\lim_{n \to \infty} 1^n = 1
\]

If \( r = -1 \)
\[ \text{diverges} \]

The sequence \( a_n = r^n \) converges if \(-1 \leq r \leq 1\) and \( \lim_{n \to \infty} r^n = 0 \) if \(-1 < r < 1\)
\[ 1 \text{ if } r = 1 \]
Some useful properties of sequences

**Definition** A sequence \(\{a_n\}\) is **increasing** if \(a_n < a_{n+1}\) for all \(n \geq 1\), so that \(a_1 < a_2 < a_3 < \ldots\). It is **decreasing** if \(a_n > a_{n+1}\) for all \(n \geq 1\). A sequence is **monotonic** if it is either increasing or decreasing.

**Examples**

1. \(\left\{\frac{2}{n+6}\right\}\) is decreasing.

\[
\frac{2}{n+6} > \frac{2}{(n+1)+6} = \frac{2}{n+7}
\]

So \(a_n > a_{n+1}\) for \(n \geq 1\).

2. Show \(a_n = \frac{n}{n^2+1}\) is decreasing.

Want \(\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}\)

\[
\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} \iff (n+1)(n+1) < n[(n+1)^2+1]
\]

\[
n^2 + 2n + 1 < n^3 + 2n^2 + n
\]

\[
1 < n^3 + n^2 \quad \text{True for}
\]

**OR** \(f(x) = \frac{x}{x^2+1}\) has \(f(n) = a_n\)

\[
f'(x) = \frac{(x^2+1) - 2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0 \quad \text{for} \quad x > 1.
\]

So \(f\) is decreasing, implying that the sequence is also decreasing.

Sequences can also be *bounded*.
Defn A sequence $\{a_n\}$ is bounded above if there is a number $M$ s.t. $a_n \leq M$ for all $n \geq 1$. It is bounded below if there is an $m$ s.t. $m \leq a_n$ for all $n \geq 1$.

If $\{a_n\}$ is bounded above and below, then $\{a_n\}$ is a bounded sequence.

Examples
- $\{2n^3\}$ bounded below but not above
- $\{\frac{n}{n+1}\}$ bounded $0 < a_n < 1$

Suppose a sequence is both bounded and monotonic.

The sequence keeps increasing but runs out below $M$. So it must converge to a limit before that.

Thm Every monotone bounded sequence is convergent.

Proof The real numbers have a property called the least upper bound property: any set $S \subseteq \mathbb{R}$ which is bounded has some $b$ such that $s \leq b$ for all $s \in S$ and there is no $c < b$ with the same property.

Let $\{a_n\}$ be bounded and monotone increasing. Then there is some $n$ s.t. $L - \varepsilon < a_n \leq L$. But $a_n$ is increasing, so $|L - a_n| < \varepsilon$ for all $n > N$. Ergo $L$ is in fact the limit of the sequence.

Similarly for decreasing sequences and greater or lower bounds.