7.3.3. **The IBVP: Neumann Conditions.**

Theorem 7.3.3. The BVP for the one-space dimensional heat equation,

$\partial_t u = k\, \partial_x^2 u$, BC: $\partial_x u(t,0) = 0$, $\partial_x u(t,L) = 0$, Neumann BC.

where $k > 0$, $L > 0$ are constants, has infinitely many solutions

$$u(t,x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n\, e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right), \quad c_n \in \mathbb{R}.$$

Furthermore, for every continuous function f on $[0, L]$ satisfying $f'(0) = f'(L) = 0$, there is a unique solution u of the boundary value problem above that also satisfies the initial condition

$$u(0,x) = f(x).$$

This solution u is given by the expression above, where the coefficients ____ are

$$c_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx.$$

Remarks:

(a) This is an Initial-Boundary Value Problem (IBVP).

(b) The boundary conditions are called Neumann boundary conditions.

Remark: The physical meaning of the initial-boundary conditions is simple.

(1) The boundary conditions is to keep the heat flux at the sides of the bar is constant.

(2) The initial condition is the initial temperature on the whole bar.

Remark: One can use Dirichlet conditions on one side and Neumann on the other side. This is called a mixed boundary condition.

Remark: The proof is based on the separation of variables.

Proof of the Theorem: $\partial_t u = k\,\partial_x^2 u$, NBC: $\boxed{\begin{aligned}&\partial_x u(t,0)=0\\&\partial_x u(t,L)=0\end{aligned}}$

Separation of variables.

Simple Solutions

$$u(t,x) = V(t)\,W(x) \longrightarrow \boxed{\begin{aligned}&(\partial_x u)\big|_{(t,\,x=0)} = 0\\&(\partial_x u)\big|_{(t,\,x=L)} = 0\end{aligned}}$$

$$\partial_t\big(V(t)\,W(x)\big) = k\,\partial_x^2\big(V(t)\,W(x)\big)$$

$$(\partial_t V)\,W(x) = k\,V(t)\,(\partial_x^2 W)$$

$$\left(\begin{aligned}\partial_t V &= \dot V\\ \partial_x W &= W'\end{aligned}\right)$$

$$\left(\dot V\,W = k\,V\,W''\right)\frac{1}{VW}\,\frac{1}{k}$$

$$\partial_x(VW)\big|_{x=0,t} = 0 = V(t)(\partial_x W)(0)$$

$$\partial_x(VW)\big|_{x=L,t} = 0 = V(t)(\partial_x W)(L)$$

$$\frac{1}{k}\,\frac{\dot V(t)}{V(t)} = \frac{W''(x)}{W(x)} = -\lambda \quad \text{constant.} \qquad \lambda \in \mathbb{R}$$

$t \qquad x$

$$\boxed{\frac{1}{k}\,\frac{\dot V}{V} = -\lambda}$$

$$\dot V(t) = -k\lambda\,V(t)$$

$$\boxed{V(t) = c\,e^{-k\lambda t}}$$

$c = V(0)$ arbitrary

$$\boxed{\frac{W''}{W} = -\lambda} \qquad \boxed{\begin{aligned}W'(0)&=0\\W'(L)&=0\end{aligned}}$$

Eigenfunction Problem.

$$\boxed{W'' + \lambda W = 0}$$

Non-zero Sols. only for $\lambda > 0$

$\lambda = \mu^2$, $\mu > 0$

$$W'' + \mu^2 W = 0 \implies r^2 + \mu^2 = 0 \implies$$

Proof of the Theorem: $\partial_t u = k\, \partial_x^2 u$, NBC: $\boxed{\begin{matrix}\partial_x u(t,0)=0 \\ \partial_x u(t,L)=0\end{matrix}}$

Separation of variables.

Simple Solutions

$u(t,x) = V(t)\, W(x)$ $\longrightarrow$ $\boxed{\begin{matrix}(\partial_x u)\big|_{(t,x=0)} = 0 \\ (\partial_x u)\big|_{(t,x=L)} = 0\end{matrix}}$

$\partial_t \big(V(t)\, W(x)\big) = k\, \partial_x^2 \big(V(t)\, W(x)\big)$

$(\partial_t V)\, W(x) = k\, V(t)\, (\partial_x^2 W)$ $\qquad \begin{pmatrix}\partial_t V = \dot{V} \\ \partial_x W = W'\end{pmatrix}$

$\big(\dot{V}\, W = k\, V\, W''\big)\, \frac{1}{VW}\, \frac{1}{k}$

$\partial_x(VW)\big|_{x=0,t} = 0 = V(t)(\partial_x W)(0)$

$\partial_x(VW)\big|_{x=L,t} = 0 = V(t)(\partial_x W)(L)$

$\frac{1}{k} \frac{\dot{V}(t)}{V(t)} = \frac{W''(x)}{W(x)} = -\lambda$ constant. $\lambda \in \mathbb{R}$

$t \qquad x$

$\boxed{\frac{1}{k} \frac{\dot{V}}{V} = -\lambda}$

$\dot{V}(t) = -k\lambda\, V(t)$

$\boxed{V(t) = c\, e^{-k\lambda t}}$

$c = V(0)$ arbitrary

$\boxed{\frac{W''}{W} = -\lambda}$ $\qquad \boxed{\begin{matrix}W'(0)=0 \\ W'(L)=0\end{matrix}}$

eigenfunction problem.

$\boxed{W'' + \lambda W = 0}$

Non-zero Sols. only for $\lambda > 0$

$\lambda = \mu^2$, $\mu > 0$

$W'' + \mu^2 W = 0 \implies r^2 + \mu^2 = 0 \implies$

Initial condition: $\underline{u(0,x) = f(x)}$; $\underline{f'(0) = f'(L) = 0}$

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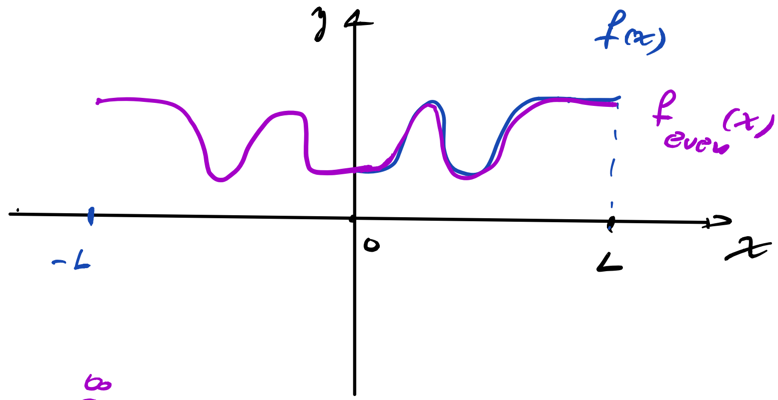
$$\boxed{f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right)}$$

Fourier Series Exp. Thrm



$\rightarrow [-L, L]$

$$f_{even}(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right)$$

$$c_n = \frac{2}{L}\int_0^L f_{even}(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad n = 0, 1, 2, \ldots$$

$f(x) = f_{even}(x)$ on $[0, L]$

$$\boxed{c_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx} \qquad n = 0, 1, 2, \ldots$$

$$\boxed{u(t,x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right)}$$

$\blacksquare$

EXAMPLE 7.3.2: Find the solution to the initial-boundary value problem

$$\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 3],$$

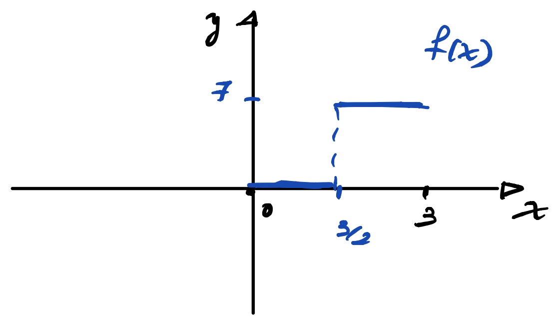
with initial and boundary conditions given by

$$\text{IC:} \quad u(0,x) = \begin{cases} 7 & x \in [\frac{3}{2}, 3], \\ 0 & x \in [0, \frac{3}{2}), \end{cases} \qquad \text{NBC:} \quad \begin{cases} u'(t,0) = 0, \\ u'(t,3) = 0. \end{cases}$$

SOLUTION:

Separation of variables.

Simple Solutions



$$u(t,x) = v(t)\, w(x)$$

$$\partial_x (v w)\big|_{(t,0)} = 0, \quad \partial_x (v w)\big|_{(t,3)} = 0 \;\Rightarrow\; \begin{cases} v(t)\, w'(0) = 0 \\ v(t)\, w'(3) = 0 \end{cases}$$

$$\dot{v}(t)\, w(x) = v(t)\, w''(x)$$

$$\frac{\dot{v}(t)}{v(t)} = \frac{w''(x)}{w(x)} = -\lambda \quad \text{const.}$$

$$\boxed{\frac{\dot{v}}{v} = -\lambda}$$

$$\dot{v} = -\lambda v$$

$$\boxed{v(t) = c\, e^{-\lambda t}}$$

$c = v(0)$, arbitrary

$$\boxed{\frac{w''}{w} = -\lambda} \quad \text{BC} \begin{bmatrix} w'(0) = 0 \\ w'(3) = 0 \end{bmatrix}$$

eigenfunction problem (Sect. 7.1)

$$\boxed{\begin{aligned} &\lambda_n = \left(\frac{n\pi}{3}\right)^2 && n = 0, 1, 2, \dots \\ &w_n(x) = \cos\left(\frac{n\pi x}{3}\right), && n = 1, 2, \dots \\ &w_0(x) = \frac{1}{2} && n = 0 \end{aligned}}$$

$$\boxed{V_n(t) = c_n \, e^{-\left(\frac{n\pi}{3}\right)^2 t}}$$

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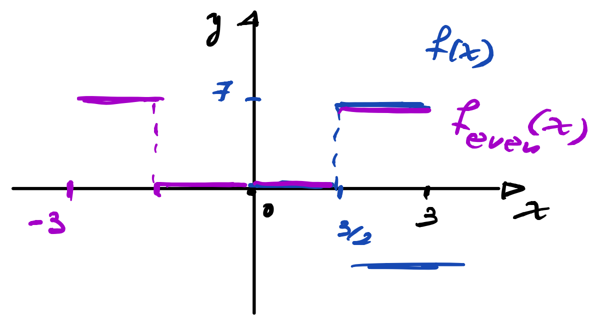
$$\boxed{u(t,x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \, e^{-\left(\frac{n\pi}{3}\right)^2 t} \cos\left(\frac{n\pi x}{3}\right)}$$ BVP

I.C. $\quad u(0,x) = f(x)$



$$f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{3}\right)$$

$$f_{even}(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{3}\right), \quad c_n = \frac{2}{3}\int_0^3 f_{even}(x) \cos\left(\frac{n\pi x}{3}\right)$$

$n = 0, 1, 2, \ldots$

$f(x) = f_{even}(x)$ on $[0, 3]$

$$c_n = \frac{2}{3}\int_0^3 f(x) \cos\left(\frac{n\pi x}{3}\right) dx$$

$$c_n = \frac{2}{3}(7)\int_{2/3}^{3} \cos\left(\frac{n\pi x}{3}\right) dx$$

$$c_0 = \frac{2}{3}(7)\int_{2/3}^{3} dx = \frac{2}{3}(7)\left(3 - \frac{2}{3}\right) = \frac{2}{3}(7)\left(\frac{9}{3} - \frac{2}{3}\right)$$

$$c_0 = \frac{2}{3}(7)\left(\frac{7}{3}\right) \Rightarrow \boxed{c_0 = 2\left(\frac{7}{3}\right)^2}$$

$$c_n = \cancel{\frac{2}{3}}(7)\,\frac{\cancel{3}}{n\pi}\left(\sin\left(\frac{n\pi x}{3}\right)\right)\Bigg|_{2/3}^{3} \qquad n=1,2,\dots$$

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$$c_n = \frac{14}{n\pi}\left(\underset{0}{\underset{\shortparallel}{\sin(n\pi)}} - \sin\left(n\pi\,\frac{2}{9}\right)\right)$$

$$\boxed{c_n = -\frac{14}{n\pi}\sin\left(\frac{2n\pi}{9}\right)}$$

$$\boxed{u(t,x) = \left(\frac{7}{3}\right)^2 - \sum_{n=1}^{\infty}\frac{14}{n\pi}\sin\left(\frac{2n\pi}{9}\right)e^{-\left(\frac{n\pi}{3}\right)^2 t}\cos\left(\frac{n\pi x}{3}\right)}$$