Review for Gateway Exam

Chapter 1

1. A population of fish has a **growth rate proportional to the amount of fish present** at that time, with a proportionality factor of $\frac{1}{5}$ per unit time.

   (a) Write a differential equation of the form $P' = F(P)$, which models this situation, where P is the number of fish as a function of time.

   (b) Now, assume that we have the same fish population, reproducing as above, but we are harvesting fish at a **constant rate of 100** fish per unit time. Write the differential equation in this case.

   (c) Assume that the initial fish population is 600 fish. Solve the ordinary differential equation in part (b) above with this given initial condition.

2. Use the Picard iteration to find the first 4 of a sequence $\{y_n\}$ of approximate solutions to the IVP

$$y'(t) = 8t^3 y(t), \quad y(0) = 4.$$

3. Find the general solution to the following ODE

$$y' - 8y^2 \cos(t) - 7y^2 \sin(4t) = 0.$$

4. A radioactive material has a **decay rate** proportional to the amount of radioactive material present at that time, with a proportionality factor of 2 per unit time.

   (a) Write a differential equation of the form $P' = F(P)$, which models this situation, where P is the amount of radioactive material (measured in micrograms) as a function of time.

   (b) Now, assume that we have the same radioactive material decaying as above, but we are **adding** additional material (of the same type) **at a constant rate of 6** micrograms per unit time. Write the differential equation in this case.

   (c) Solve the ordinary differential equation in part (b) above, assuming the initial amount of radioactive material is 70 micrograms.

5. Find the solution to the following initial value problem

$$y' + 8y^3 \cos(7t) = 0, \qquad y(0) = 2.$$

6. Find the solution to the following IVP

$$ty' = 2y - 3t^3 \cos(4t), \qquad y(\pi/8) = 0.$$

7. Consider the differential equation
$$\frac{dy}{dt} = y(y^2 - 4)(y^2 + 9).$$
   (a) Find the **equilibrium solutions** of the ODE.
   (b) Construct a phase diagram and determine the stability of the critical points.
   (c) Make rough sketches of typical solution curves.

8. Find the solution to the following IVP
$$y' = \tan(t)y - 5t, \quad t \in [0, \frac{\pi}{2}), \qquad y(0) = 3.$$

9. Find an explicit expression for the solution y of the following initial value problem.
$$y' = \frac{3y^3 + t^3}{ty^2}, \quad y(1) = 2, \quad t \geq 1.$$

10. A glass of cold soda is placed into a room held at 30 C.
   (a) If k is a (positive cooling constant), find the differential equation satisfied by the temperature, $T(t)$ of the soda.
   (b) Find the soda temperature as a function of time (and k), if the initial temperature of the soda was 2 C.
   (c) If after 40 minutes the soda temperature was 10 C, find the cooling constant k.

Answers and Hints

Problem (1): Answer for (c): $P(t) = 500 + 100\, e^{t/5}$.

Problem (2): Partial Answer: $y_3(t) = 4 + 8t^4 + 8t^8 + \frac{16}{3} t^{12}$.

Problem (3): Answer: $y(t) = \left(-8\sin(t) + \frac{7}{4}\cos(4t) + c \right)^{-1}$

Problem (4): Answer for (c): $P(t) = 3 + 67\, e^{-2t}$

Problem (5): Answer: $y(t) = \left(\frac{16}{7}\sin(7t) + \frac{1}{4} \right)^{-1/2}$

Problem (6): Answer: $y(t) = \frac{3}{4} t^2 - \frac{3}{4} t^2 \sin(4t)$

Problem (7): Answer to a portion of (b): critical pts: -2 (unstable), 0 (stable) , 2 (unstable).

Problem (8): Answer: $y(t) = \frac{8}{\cos(t)} - 5t\tan(t) - 5$

Problem (9): Answer: $y(t) = t\left(\frac{17}{2} t^6 - \frac{1}{2} \right)^{1/3}$

Problem (10): Answer: (a) $T' = -k(T - 30)$; (b) $T(t) = 30 - 28e^{-kt}$; (c) $k = \frac{1}{40}\ln\left(\frac{7}{5}\right)$